SCHALLER

General Theory of Force And Mass Reductions with Examples from Existing Machines

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GENERAL THEORY OF FORCE AND MASS REDUCTIONS WITH EXAMPLES FROM EXISTING MACHINES

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

ALWIN L SCHALLER

ENTITLED GENERAL THEORY OF FORCE AND MASS REDUCTIONS WITH EXAMPLES FROM EXISTING MACHINES

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Bachelor of Science in Mechanical Engineering

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Contents

The General Theory of Force and Mass Reduction with Examples from Existing Machines.

I - Introduction

In many problems involving the kinetics of mechanisms it is often necessary to replace all masses by an equivalent mass concentrated at some particular point and to replace all forces by equivalent forces having chosen lines of action. It is the object of this thesis to develope equations that may be employed for making such reductions and to apply these equations to some existing machine. Considerations not directly connected with the subject but having an important bearing upon its development are also taken up; As for example, a discussion of kinetic energy and the determination of inertia forces. The theory is applied to one of the fly ball governors in

the laboratory. In order to make the subject complete the derivation of the equation of motion of the fly-ball governor is included. The application has been limited to mechanisms without springs but the equations are of a general nature and may be applied without difficulty to this class of machines.

Many notations were used throughout the discussion and it was found convenient to include them here to save repeating them whenever necessary.

Notation.

M = mass.

F = force.

a = linear acceleration

g = acceleration due to gravity.

V = volume.

I = principal moment of inertia

I' = moment of inertia with respect to axis of rotation

ΔM = element of mass.

k = principal radius of gyration

T = time of oscillation of a pendulum.

v = velocity.

$\frac{d\theta}{dt}$ = angular velocity.

ω = angular velocity.

$\frac{dx}{dt}$ = linear velocity.

α = angular acceleration

$\mathcal{M}$ = reduced mass.

$\mathcal{F}$ = reduced force.

W = work.

II - Mass Reduction

1- When masses are reduced from one point of a system to another, the dynamic conditions must remain the same, that is, the velocities and accelerations of the moving masses must remain unchanged by the reduction. In order that this condition may obtain, the reduced system at any instant must be equal to the total kinetic energy of all of the separate masses moving at that instant.

2- Mass, Moment of Inertia, etc. - Before passing to the methods of reducing masses, a general discussion of mass and its properties will be given. By the term, mass, is meant the amount of matter in a body, independent of place. Mass is, therefore, distinct from weight which is merely the attraction due to gravity and varies

from place to place. For measuring mass, Newton's Second Law of Motion is employed. "The acceleration of a body is directly proportional to the force applied and inversely to the mass." This law stated in the form of an equation is

$$F \propto Ma \quad \text{or} \quad F = kMa$$

where k = constant. The value of the constant k depends upon the units employed for expressing the remaining terms of the equation. The simplest system of units would be one in which k would reduce to 1 (or unity) and the equation would then read

$$F = Ma.$$

In the Engineers System (f.,p.,s.) the units are foot, pound and second. The pound used here is a unit of force. From these units, the unit of acceleration at once is ft./sec.^{2}. It now remains to deduce such a unit of mass as to make k equal to 1. Consider a body under the action of gravity of weight W. The force act-

ing in this case is simply the weight. If the acceleration is denoted by g, we have therefore

$$W = Mg \text{ or } M = \frac{W}{g} \text{ (If } M \text{ is in proper units)}$$

We now have the important relation that the mass is equal to the weight divided by the acceleration due to gravity. The unit of mass then is

$$\frac{F}{ft/sec^2} = \frac{F\ sec^2}{ft.}$$

Having found the units we can check our equations by substituting in them, these fundamental units and noting whether both members are of the same kind.

$$F = Ma$$
$$p = \frac{p\ sec^2}{ft.} \times \frac{ft.}{sec^2}$$
$$p = p.$$

We have now to find a numerical value for g. This has been done experimentally both with falling bodies and with the pendulum. An average value is $32.2 \frac{ft.}{sec^2}$. The Engineers System of units will be em-

ployed throughout this discussion.

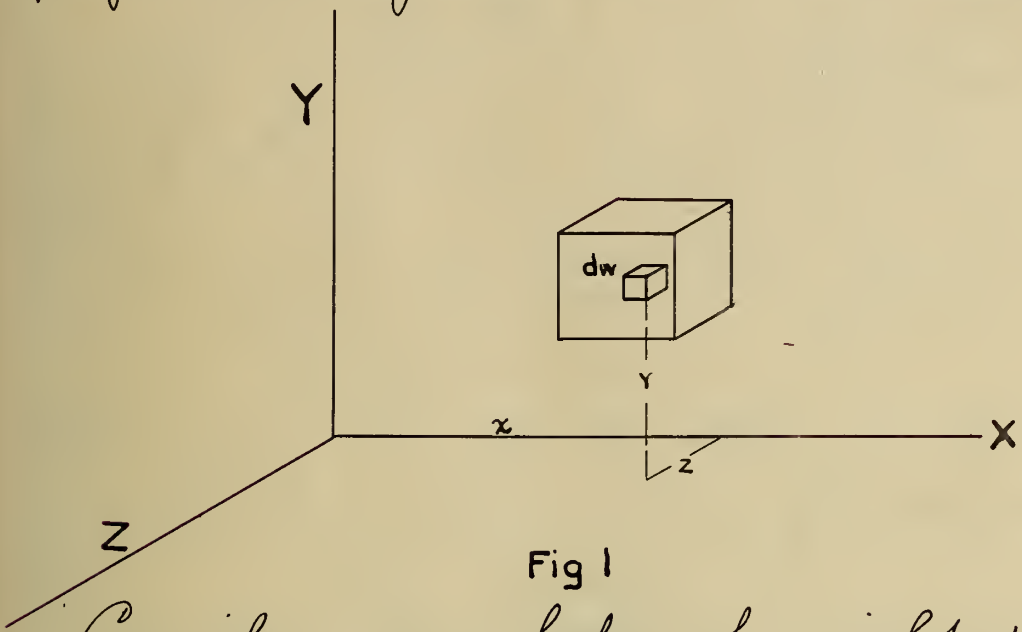


Fig 1

Consider any body of weight W as shown in figure 1. Take any particle, dW, whose coordinates are x, y and z. Gravity acts upon this element with a force equal to the weight of the element. The weights of all the elements of the body form a system of parallel forces. The resultant of these forces passes through a fixed point with reference to the body independent of its position. This point is called the center of gravity. To locate this point, use is made of the following principle. The weight of a body multiplied by the distance of its

center of gravity from any datum plane is equal to the sum of all the elements of that body multiplied by their coordinates with respect to the same plane. The moment of any particle with respect to the YZ plane is

$$dM_{yz} = x\,dw$$

and if $\bar{x}$ is coordinate of the center of gravity from the same plane, we have

$$W\bar{x} = \Sigma x\,dw$$

$$\bar{x} = \frac{\Sigma x\,dw}{W} \qquad (1)$$

The limits of the summation are such as to include the bounding surfaces of the body. In the same way, we have

$$\bar{y} = \frac{\Sigma y\,dw}{W}$$

$$\bar{z} = \frac{\Sigma z\,dw}{W}$$

The center of gravity may therefore be determined by integration when the equation of the bounding surface is given.

Let K = weight per unit volume

Then $VK = W$ and $K\,dV = dw$

Substituting in (1)

$$\bar{x} = \frac{K\int x\,dv}{KV} = \frac{\int x\,dv}{V}$$

This equation may now be integrated when the limits are known.

Most bodies with which we have to deal, however, are of such irregular outline that no equations can be given for the bounding surfaces. Any machine link will serve for illustration. In such cases, the center of gravity must be determined experimentally. In fact, the given equation is of greater value in defining the center of gravity and gaining a clear conception of what it is than in actually locating it in machine links. One method of finding the center of gravity is to suspend the body by a cord. The direction of the cord is then marked on the body, and the operation repeated with a different point of suspension. The center of gravity will lie in each one of these lines and evidently at the point of their intersection.

Figure 2 illustrates this procedure.

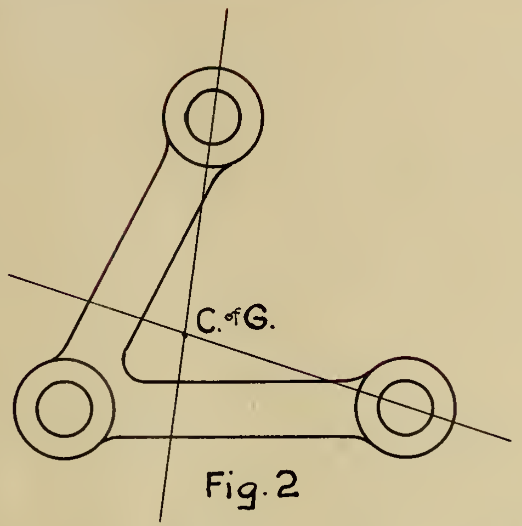


Fig. 2

The second method is to balance the body on a knife edge, the vertical plane of which is indicated on the body. This operation is repeated for two other positions; and, as the center of gravity lies in the vertical plane of the straight edge, it is in the point of intersection of the three planes. Another method is to rest one end of the body on a fixed knife edge and suspend the other with a spring balance, as shown in figure 3.

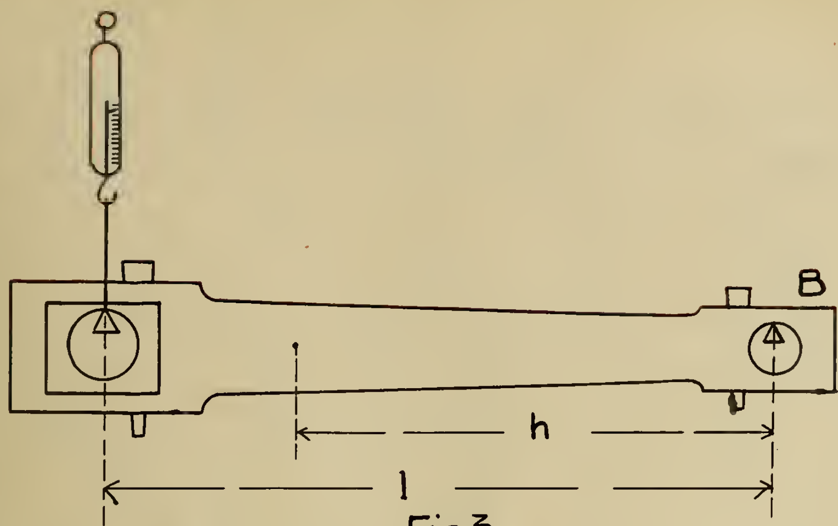


Fig 3

Let F = force on scale

W = weight of link

l = distance between knife edges.

Taking moments about B, we have.

$$Fl = Wh \quad \text{or} \quad h = \frac{Fl}{W}$$

In a preceding paragraph it was shown that the weight and mass of a body had a fixed relation, viz:

$$M = \frac{W}{g}$$

Substituting Mg for W in the following equation

$$\bar{x} = \frac{\Sigma x\, dW}{W} = \frac{g\Sigma x\, dM}{Mg} = \frac{\Sigma x\, dM}{M}$$

This proves the important relation that the center of mass coincides with the center of gravity.

In the study of rotating masses, the expression $\Sigma r^2 dM$ is often met with. To this expression Euler gave the name, moment of inertia. The definition of moment of inertia of a body is therefore the sum of the products obtained by multiplying each element of mass by the square of its distance from a fixed axis.

To find the moment of inertia, proceed as follows:

Let ρ = distance of any element.

γ = density

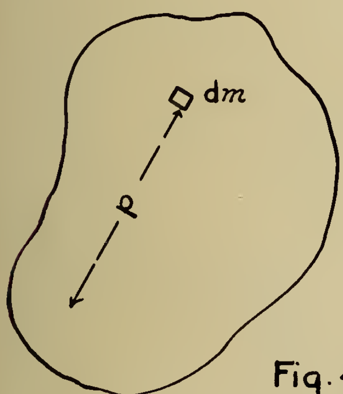


Fig. 4.

By definition

$$I = \Sigma\, dM \rho^2$$

But $dM = \gamma\, dV$

$$\therefore I = \gamma \Sigma\, dV \rho^2$$

The limits of integration must be such so as to include all elements of the body.

We may express the moment of inertia as the product of the whole mass by the square of a length as

$$MK^2 = I \quad \text{or} \quad K = \sqrt{\frac{I}{M}}$$

This length is known as the radius of gyration.

An example will now be given illustrating the method of finding the moment of inertia of a body by integration.

Let r = radius of base of a cylinder

h = altitude

γ = density

dV = element of volume

ρ = distance of the element from the axis

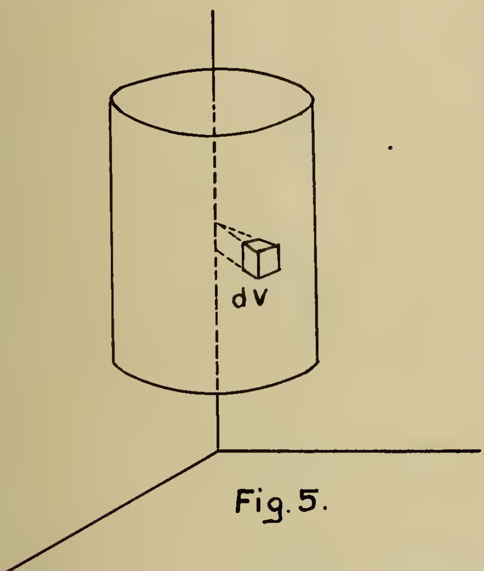


Fig. 5.

Then

$$I = \gamma \int dV \rho^2$$

$$dV = \rho \, d\theta \, d\rho \, dh$$

$$I = \gamma \int_0^r \int_0^{2\pi} \int_0^h \rho^3 \, d\theta \, d\rho \, dh$$

$$= h\gamma \int_0^r \int_0^{2\pi} \rho^3 \, d\theta \, d\rho = 2\pi h \gamma \int_0^r \rho^3 \, d\rho$$

$$= \frac{\pi h r^4 \gamma}{4} \qquad \text{But } \pi r^2 h \gamma = M$$

$$\therefore I = \frac{M}{2} r^2 \qquad k^2 = \frac{r^2}{2}$$

It is evident that these expressions will always be in their simplest form,

when the axis of inertia is the centroidal axis of the body. If the moment of inertia is desired for any parallel axis it is usually reduced from this centroidal axis. An expression will now be given by means of which this reduction may be effected.

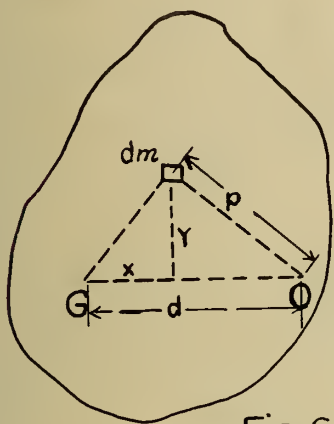


Fig. 6.

Let Fig. 6 show section of a body perpendicular to the inertia axis. Let G and O be points where the inertia and parallel axes pierce the plane of the section.

Let $\bar{I}$ = Moment of inertia with respect to axis G.

I = " " " " " " " O

d = Distance between O and G.

$$I = \int dM \rho^2$$

$$\rho^2 = y^2 + (d-x)^2$$

$$\therefore I = \int dM \left[y^2 + d^2 - 2dx + x^2\right]$$

$$= \int dM(y^2 + x^2) + \int dM d^2 - 2\int dM dx$$

But $\int dM(x^2+y^2) = \bar{I}$

$$\int dM d^2 \quad = Md^2$$

$$-2\int dM\, dx = 0 \quad (G \text{ is center of gravity})$$

$$\therefore\ I = \bar{I} + Md^2$$

Dividing by M, we have

$$\frac{I}{M} = \frac{\bar{I}}{M} + \frac{Md^2}{M} \text{ or } K^2 = \bar{K}^2 + d^2$$

These equations are valuable in obtaining the moments of inertia of a composite body, for example, a fly wheel. By integration or by general formulae, the moment of inertia of the rim may be found, considering it as a hollow cylinder. Next the moment of inertia of the arms are found about a centroidal axis and then reduced to the axis of the shaft. Finally, these quantities are all added together and the result is the moment of inertia of the entire wheel.

But, as in the case of finding the center of gravity, the bodies with which we have to deal are for the most part of such shapes that experimental rather than analytical methods must be adopted. There are several ways

in which we may find the moment of inertia of irregular bodies, but the two in general use are the pendulum method and the tortion balance method.

(a) The pendulum method.

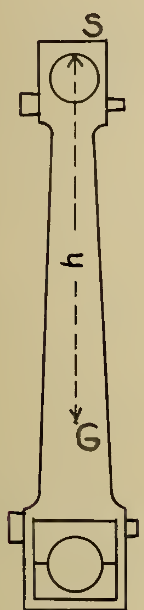


Fig. 7.

Suspend link as shown and allow it to oscillate, observing carefully the time, T. The equation of the pendulum is

$$T = 2\pi\sqrt{\frac{k^2}{hg}} \quad \text{(Maurer's Mech.)}$$

or $k = \frac{T}{2\pi}\sqrt{hg}$

h = distance from point of suspension to center of gravity.

All quantities being known, k can be readily calculated. The principal radius of gyration can be found by the equation already given.

(b) - Torsion Balance.

A flat plate is suspended by an elastic wire which is firmly fastened to some support. The body

whose moment of inertia is to be found is laid upon the plate along a second body so as to leave the plate in a horizontal position. The position of the first is such that the inertia axis is parallel to the wire, and that of the second such that its I with respect to the wire is known. Let the balance oscillate, and observe the time, T; remove the bodies and let the empty balance oscillate again and observe the time, T'.

Let I_1 = I of plate with respect to wire
I_2 = " " second body " " "
I = " " link " " " "

From equation of balance

$$T = \pi\sqrt{\frac{I}{c}} \qquad \text{(Maurer's Mechanics)}$$

we have that the I's are proportional to the squares of the time or

$$T : T' :: \sqrt{I_1 + I_2 + I} : \sqrt{I}$$

or
$$I = \frac{I_1 T^2}{T'^2} - [I_1 + I_2]$$

Concentrated Mass - When a body is considered to be concentrated at one

point, this consideration implies that it has no volume. Therefore, its moment of inertia with respect to an axis through the point is zero, and with respect to any other axis is simply Md^2 when M is the mass and d the distance between the axis and the point where the mass is considered to be concentrated.

3- Kinetic Energy - A body possesses energy when it is capable of doing work in changing its state. Energy which is due to velocity of body is known as kinetic energy. As work can be directly converted into energy and energy into work, these quantities must be expressed in the same units, which in the Engineer's System are foot-pounds.

The fundamental equation for kinetic energy is derived from the relation that energy is equal to work done.

Work is product of force and distance.

$$W = \int_0^s F \cdot ds = E_K$$

But $F = \Delta M a = \Delta M \frac{dv}{dt}$

$$E_K = \int_0^v \Delta M \frac{dv}{dt}\, ds = \int_0^v \Delta M \cdot v\, dv$$

$$= \frac{1}{2} \Delta M v^2$$

A body may be considered as made up of a system of particles. For translation, the velocity of all particles is the same. Then

$$E_K = \Sigma \tfrac{1}{2} \Delta M v^2$$

$$= \frac{1}{2} M v^2$$

For a rotating system, the velocity of any particle is $r\omega$. Then

$$E_K = \Sigma \tfrac{1}{2} \Delta M (r\omega)^2$$

$$= \frac{\omega^2}{2} \Sigma \Delta M r^2$$

$$= \frac{1}{2} I' \omega^2$$

This equation has a wider application than the first for it may be applied to a body with any plane motion, for a body may be considered

as rotating about its instantaneous center. In this case the moment of inertia would in general be constantly varying as the instantaneous center of rotation is not fixed. The kinetic energy may then be divided into two parts, one due to a rotation about the center of gravity, and the second due to the translation of the center of gravity itself. We have therefore

$$I' = \bar{I} + Mr^2$$

$$E_K = \frac{1}{2} I'\omega^2 = \frac{1}{2}\omega^2\left[I + Mr^2\right]$$

$$= \frac{1}{2}\omega^2 I + \frac{1}{2}Mv^2$$

4 - Equations for Reducing Masses - General expressions which can be used for reducing the mass of a body to any point will now be deduced. The following notation will be used in all subsequent work. Original masses will be denoted by roman capitals with suitable subscripts, reduced masses with corresponding script capitals.

The first case to be considered

is the reduction of a rotating mass to any point in the same plane of rotation. The principle for reducing masses has already been enunciated, namely, that the kinetic energy of the reduced mass at any instant must equal the total energy of all separate masses moving at that instant.

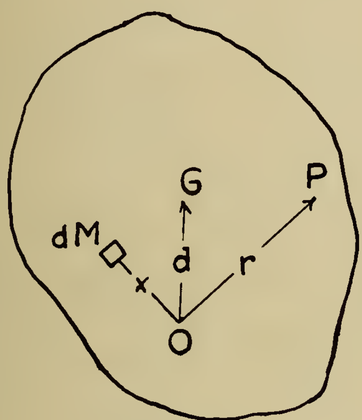


Fig. 8.

Let O = center of rotation
P = point to which mass is to be reduced
G = center of gravity
r = distance from O to P

Take any differential element of mass, dM, at a distance x from O. Let $d\mathcal{M}$ be corresponding element reduced to point P.

The kinetic energy of a particle dM

$$= \frac{1}{2} dM\, x^2 \left(\frac{d\theta}{dt}\right)^2$$

The kinetic energy of particle $d\mathcal{M}$

$$= \frac{1}{2} d\mathcal{M}\, r^2 \left(\frac{d\theta}{dt}\right)^2$$

Hence $\frac{1}{2} dM\, x^2 \left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2} d\mathcal{M}\, r^2 \left(\frac{d\theta}{dt}\right)^2$

$$\int dM x^2 = r^2 \int dM$$

But $\int dM x^2 = I'$

therefore $\quad I' = M r^2$

or
$$M = \frac{I'}{r^2} = \frac{I + Md^2}{r^2}$$

This equation stated in words is as follows: <u>The reduced mass at any point is equal to the moment of inertia divided by the square of the distance from the center of rotation to that point.</u> The equation may be used for solving various problems relating to rotating bodies, and in many cases the solution is thus considerably simplified. A few examples will serve to illustrate the application.

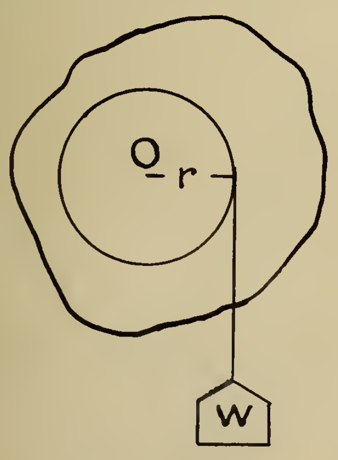


Fig. 9.

The body shown in figure 9 is made to rotate by a weight W, which is fastened to a cord wrapped about a cylindrical portion of the body

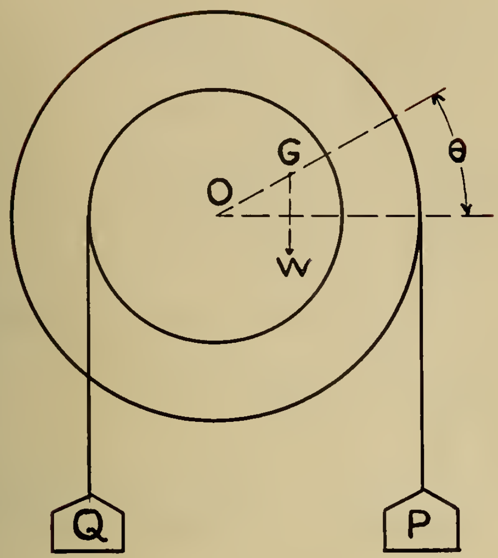


Fig. 10.

In figure 10, let
W = weight of rotating body.
$\bar{r}$ = radius of center of gravity
r = radius of Q
R = radius of P
a = acceleration of P

W reduced to radius of $P = \frac{I'}{R^2}$.

Total mass moved $= \frac{I'}{R^2} + \frac{P}{g} + \frac{Qr}{Rg}$

Forces acting are P, Q and $W\bar{r}\cos\theta$, therefore the total force at radius R

$$= P + \frac{W\bar{r}\cos\theta}{R} - \frac{Qr}{R}$$

We now have

$$P + \frac{W\bar{r}\cos\theta}{R} - \frac{Qr}{R} = \left[\frac{I'}{R^2} + \frac{P}{g} + \frac{Qr}{Rg}\right]a$$

or

$$a = \frac{PR + W\bar{r}\cos\theta - Qr}{\left[\frac{I'}{R} + \frac{PR}{g} + \frac{Qr}{g}\right]}$$

The acceleration for any given angle of θ can now be computed.

of radius r. It is required to find the acceleration of the weight and of the rotating body.

Reduce the rotating mass to a point on surface of the cylinder.

$$M = \frac{I'}{r^2}$$

The total mass moved is $\frac{I'}{r^2} + \frac{W}{g}$

The force acting is the weight W

But $F = Ma$

$$\therefore \quad W = \left[\frac{I'}{r^2} + \frac{W}{g}\right] a$$

$$a = \frac{W}{\left[\frac{I'}{r^2} + \frac{W}{g}\right]} = \frac{Wg}{\frac{Ig}{r^2} + W}$$

$$a = r\alpha$$

$$\therefore \quad \alpha = \frac{Wg}{\frac{I'g}{r} + Wr}$$

The advantage of using reduced masses in solution of the preceding problems is now evident. The problems can be solved at once from one equation. If the method of moments is used, the acceleration is found by solving simultaneous equations with three or more variables.

The second case is the reduction of a mass having a motion of translation to a point of a rotating system.

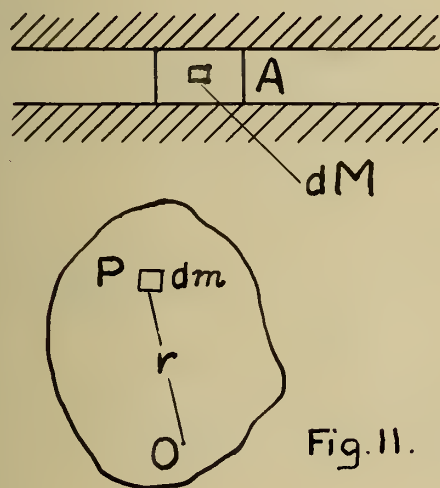


Fig. 11.

In figure 11, let

A = translating system
B = rotating system
O = center of rotation
P = point to which A is to be reduced

Kinetic energy of particle, $dM = \frac{1}{2} dM \left(\frac{dx}{dt}\right)^2$

Kinetic energy of particle, $d\mathcal{M} = \frac{1}{2} d\mathcal{M}\, r^2 \left(\frac{d\theta}{dt}\right)^2$

$$\therefore \frac{1}{2} dM \left(\frac{dx}{dt}\right)^2 = \frac{1}{2} d\mathcal{M}\, r^2 \left(\frac{d\theta}{dt}\right)^2$$

or $$\mathcal{M} = \frac{M}{r^2} \frac{\left(\frac{dx}{dt}\right)^2}{\left(\frac{d\theta}{dt}\right)^2}$$

Befor this expression can be determined the relation between $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$ must be known. This relation is given by the geometry of the figure. Take, for example, the familiar slider crank mechanism shown in figure

Fig. 12

Let r = radius of crank.
l = length of connecting rod.
θ = angle of crank.

It is required to reduce the reciprocating mass M to the radius of the crank. From preceding equation it is evident that the velocity of the crosshead must be found.

From figure

$$OS = X = r + l - r\cos\theta - \sqrt{l^2 - r^2\sin^2\theta}$$

$$\frac{dx}{dt} = r\sin\theta\,\frac{d\theta}{dt} + \frac{r^2\sin\theta\cos\theta}{\sqrt{l^2 - r^2\sin^2\theta}}\cdot\frac{d\theta}{dt}$$

But $\frac{d\theta}{dt} = \omega$ the angular velocity of the crank.

Therefore $\frac{dx}{dt} = r\omega\left[\sin\theta + \frac{\sin\theta\cos\theta}{\left(\frac{l^2}{r^2} - \sin^2\theta\right)^{\frac{1}{2}}}\right]$

Now $\mathcal{M} = \frac{M}{r^2\omega^2}\left(\frac{dx}{dt}\right)^2$

Therefore $\mathcal{M} = M\left[\sin\theta + \frac{\sin\theta\cos\theta}{\left(\frac{l^2}{r^2} - \sin^2\theta\right)^{\frac{1}{2}}}\right]^2$

This expression is often simplified by the introduction of approximations which are allowable when the ratio of the connecting rod to the crank is large. This condition is obtained when the mass of the valve and valve rod is reduced to the radius of the eccentric.

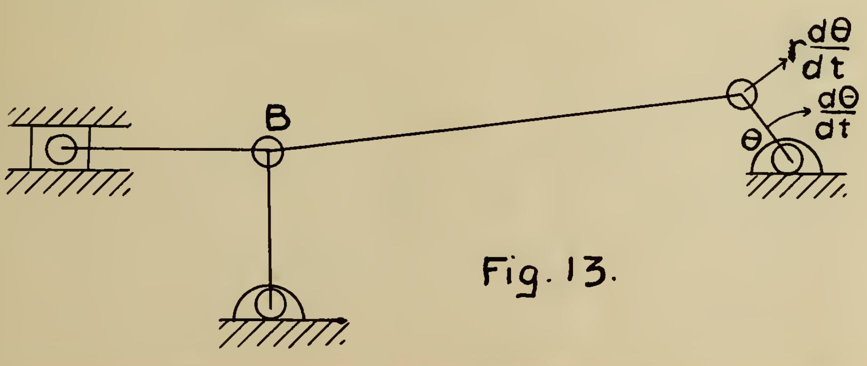


Fig. 13.

Linear velocity of A $= r\frac{d\theta}{dt}$.
Velocity of B is approximately
$r\frac{d\theta}{dt}\sin\theta$

Therefore $\mathfrak{M} = \frac{M\left(\frac{dx}{dt}\right)^2}{r^2\left(\frac{d\theta}{dt}\right)^2} = \frac{Mr^2\left(\frac{d\theta}{dt}\right)^2\sin^2\theta}{r^2\left(\frac{d\theta}{dt}\right)^2}$

$$\mathfrak{M} = M\sin^2\theta.$$

The last case to be considered is the reduction of the mass of any link of a machine to some point on another link.

I'_{in} = moment of inertia with respect to its instantaneous center

i_n = instantaneous center of link $\underline{n}$.

ω_{in} = angular velocity about instantaneous center.

i_p = instantaneous center of $\underline{p}$.

E = point on link $\underline{p}$ to which $\underline{n}$ is to be reduced.

r = radius of E from i_p.

ω_{pi} = angular velocity of link $\underline{p}$.

$$I'_{in} = I + M[in - G_{in}]^2$$

Kinetic energy of link $\underline{n} = \frac{1}{2} I'_{in} \omega^2_{in}$

Hence kinetic energy of reduced mass $= \frac{1}{2} \mathcal{M} r^2 \omega^2_{p_1}$

$$\frac{1}{2} \mathcal{M} r^2 \omega^2_{pi} = \frac{1}{2} I'_{in} \omega^2_{in} \quad \text{or}$$

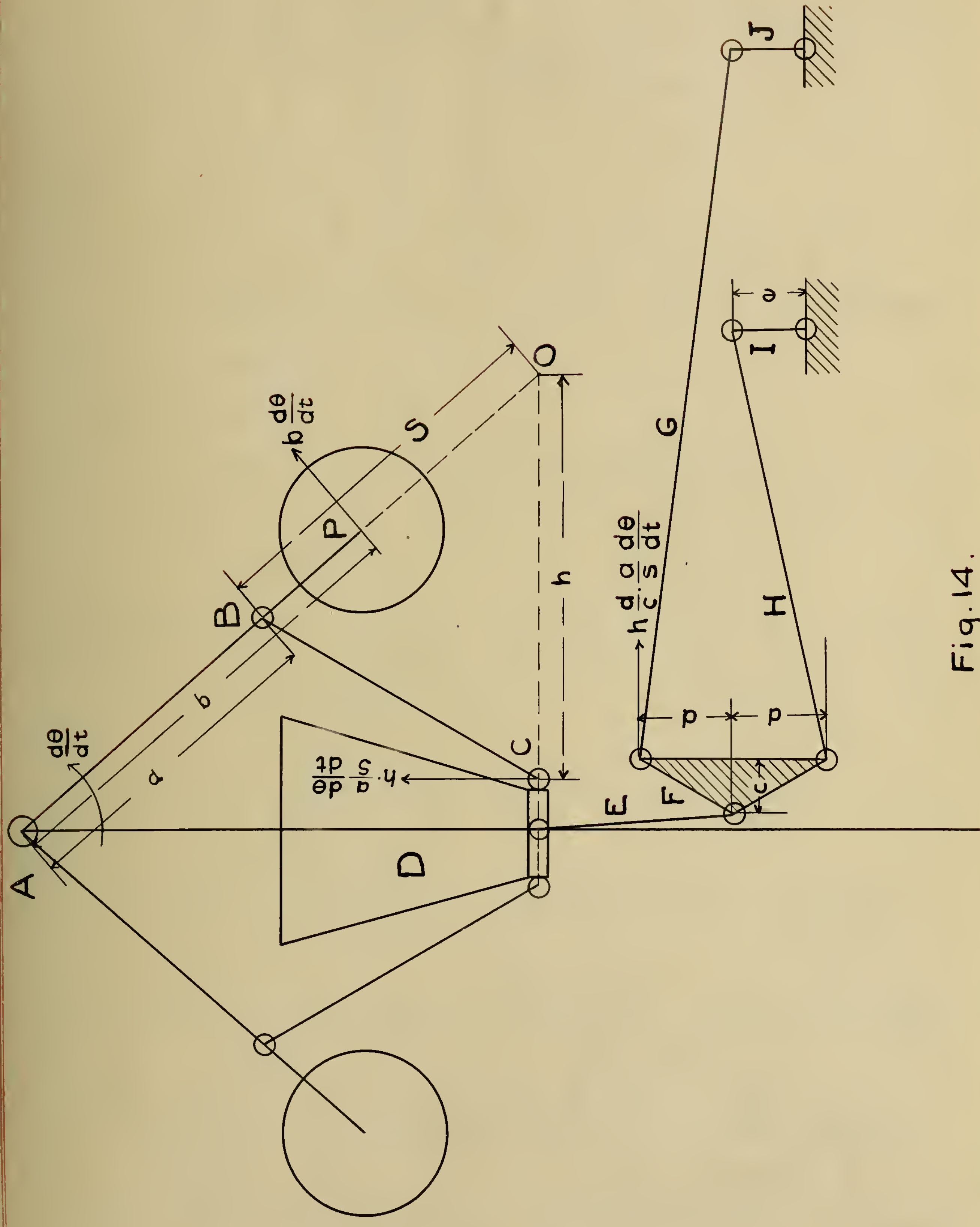
$$\mathcal{M} = \frac{I'_{in} \omega^2_{in}}{\omega^2_{pi} r^2} = \frac{M[K^2 + (in - G_{in})^2]}{r^2} \frac{\omega^2_{in}}{\omega^2_{pi}}.$$

To illustrate the use of these expressions just derived we will take an existing flyball governor and reduce all masses to the center of the ball. Two conditions should be carefully noted. The motion of the governor about its own axis is not considered. The fly weight is given a slight movement and the kinetic energy of all of the links is that

due to the velocity which is obtained by this motion. It should also be observed that for every expression derived for reduced mass one of the factors always is the ratio of the velocities. This shows that the rate which the mechanism is moving does not affect the result, because the ratio of the velocities of the two links depends only upon the geometry of the figure.

Figure 14 is a sketch of the governor on the York Ice Machine in the Mechanical Engineering Laboratory

Let I = moment of inertia of link ABP with respect to center of rotation.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B
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$\frac{d\theta}{dt}$
$b\frac{d\theta}{dt}$
$h\cdot\frac{a}{S}\frac{d\theta}{dt}$
$h\frac{d}{c}\cdot\frac{a}{S}\frac{d\theta}{dt}$

Fig. 14.

I_1 = moment of inertia of link CB with respect to instantaneous center of rotation.

M_1 = mass of D.

M_2 = mass of E.

I_2 = moment of inertia of F with respect to center of rotation.

M_3 = mass of G.

M_4 = mass of H.

I_3 = moment of inertia of I and J with respect to center of rotation.

The governor has an angular velocity $\frac{d\theta}{dt}$ about the pin A

Linear velocity of P $= b\frac{d\theta}{dt}$.

Kinetic energy of ABP $= \frac{1}{2} I\left(\frac{d\theta}{dt}\right)^2$

Kinetic energy of $d\mathcal{M} = dm\,\frac{b^2}{2}\left(\frac{d\theta}{dt}\right)^2$.

Therefore $\frac{1}{2} I\left(\frac{d\theta}{dt}\right)^2 = \sum dm\,\frac{b^2}{2}\left(\frac{d\theta}{dt}\right)^2$.

Or $\mathcal{M} = \frac{I}{b^2}$ = mass of link ABP reduced to P.

Link CB rotates about instantaneous center O. Linear velocity of B is $a\frac{d\theta}{dt}$, therefore angular velocity about O is $\frac{a}{s}\frac{d\theta}{dt}$.

Kinetic energy $= \frac{1}{2} I_1 \left(\frac{a}{s}\frac{d\theta}{dt}\right)^2$
And as before

$$\frac{1}{2} I_1 \left(\frac{a}{s}\frac{d\theta}{dt}\right)^2 = M_2 \frac{b^2}{2}\left(\frac{d\theta}{dt}\right)^2$$

$$M_2 = \frac{I_1 a^2}{s^2 b^2}$$

Point C has linear velocity $h\frac{a}{s}\frac{d\theta}{dt}$ which is also velocity of D and a close approximation to that of E. We shall therefore consider D and E as one body. We have therefore directly

$$M_3 = \frac{M_1 + M_2\left(h\frac{a}{s}\frac{d\theta}{dt}\right)^2}{b^2\left(\frac{d\theta}{dt}\right)^2} = M_1 + M_2\left(\frac{h}{b}\cdot\frac{a}{s}\right)^2$$

The angular velocity of F is $\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}$ and its kinetic energy is

$$\frac{I_2}{2}\left[\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}\right]^2 \text{ or}$$

$$\frac{I_2}{2}\left[\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}\right]^2 = M_4\,\frac{b^2}{2}\left(\frac{d\theta}{dt}\right)^2$$

$$M_4 = I_2\left[\frac{1}{b}\cdot\frac{h}{c}\cdot\frac{a}{s}\right]^2$$

The motion of the rods G and H may be taken as a translation. The velocity is $d\cdot\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}$

Therefore $M_5 = \dfrac{M_3\left[d\cdot\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}\right]^2}{\left[b\cdot\frac{d\theta}{dt}\right]^2}$

$$M_5 = M_3\left[\frac{d}{b}\cdot\frac{h}{c}\cdot\frac{a}{s}\right]^2$$

In the same way

$$M_6 = M_4\left[\frac{d}{b}\cdot\frac{h}{c}\cdot\frac{a}{s}\right]^2$$

Angular velocity of link 1 is $\frac{d}{l}\cdot\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}$ and its kinetic energy is

$$\frac{I_3}{2}\left[\frac{d}{l}\cdot\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}\right]^2$$

Therefore

$$\frac{M_7}{2}\,b\left(\frac{d\theta}{dt}\right)^2 = \frac{I_3}{2}\left[\frac{d}{l}\cdot\frac{h}{c}\cdot\frac{a}{s}\cdot\frac{d\theta}{dt}\right]^2$$

$$M_7 = I_3\left[\frac{1}{b}\cdot\frac{d}{l}\cdot\frac{h}{c}\cdot\frac{a}{5}\right]^2$$

as I and J are of the same size and have the same velocity the last expression also gives the mass of J reduced to P.

The addition of the quantities just obtained gives the reduced mass of all links of the governor that accumulate kinetic energy. It is at once evident that this sum varies for each configuration of the mechanism. If this reduced mass were calculated for several different positions and a perpindicular erected upon a straight line base directly below the point P and equal to the quantity thus obtained a smooth curve through these points would give the reduced mass curve. When this curve is once obtained the reduced mass for any position of the governor

can be scaled from the curve.

In the preceeding discussion analytical methods were used entirely simply for the sake of clearness. For complicated mechanisms however, the expressions for velocities become so long and involved that it is necessary to substitute graphical methods. This may be done by means of instantaneous centers or velocity polygons. The latter method gives an elegant solution because when a velocity polygon has been constructed it gives the velocity of every point of the mechanism.

III - Force Reduction.

1. Forces arising which may be reduced.

Whenever a mechanism changes its configuration, energy is imparted to it by forces arising from the accelerations of the moving masses. In order to effect a reduction of these forces to some point the following condition must hold. The energy imparted to the reduced system must be the same as that imparted to the actual system. In other words, the work of the reduced force for any interval of time must be equal to the sum of the works performed by the actual driving forces for the same interval.

Force is usually defined as the action of one body upon another that tends to change motion, either in direction or magnitude. In fact the term

force was introduced to account for the observed changes of motion of bodies. Our first conception of force is founded on our experience with forces exerted by ourselves.

The first force to be considered acting upon the links of a machine is the force of gravity or simply the weight. The reduction of this force should not be confused with the reductions of the masses, given in a previous paragraph. The remaining forces which arise are due to the accelerations of the moving masses. Let us first consider the simplest case, namely, that of a body rotating about a fixed center.

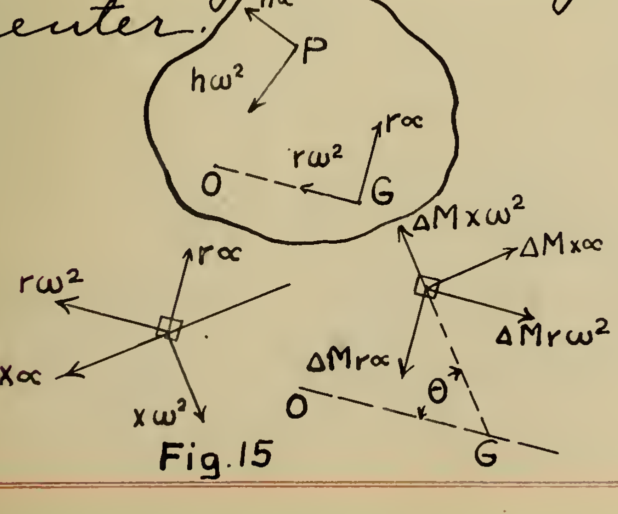

Fig. 15

In figure 15 let
P = any point
G = the center of gravity.
O = the center of rotation.

r = the radius of G
h = the distance from O to P.
x = the distance from G to P.

The acceleration at P may be divided into two components; the tangential and the normal. The value of the former is $h\alpha$ and that of the latter $h\omega^2$. For convenience of summation, the rotation about O may be replaced by a rotation about G with an equal angular velocity ω, and a translation which must be perpindicular to OG. The acceleration components of P will now be the acceleration due to the rotation about G and the acceleration due to translation which is common to all points of the body. These components are also shown in the figure.

Let ΔM denote an element of mass at point P; the accelerating forces at P are found

by multiplying the acceleration components by ΔM. By D'Alemberts principal these forces reversed in sense will hold the external forces acting on ΔM in equilibrium. The reversed forces are shown in the figure. We will now sum each of these forces for the entire body.

$$\Sigma r\omega^2 \Delta M = Mr\omega^2 = F_r$$

The forces $\Delta M r\omega^2$ form a system of parallel and equal forces, therefore they have a single resultant which passes through the center of gravity. The forces $r\alpha\Delta M$ are all equal and perpendicular to OG. They therefore have the single resultant.

$$\Sigma r\alpha\Delta M = Mr\alpha = F_t.$$

To find the resultant of forces

$x\omega^2 \Delta M$ resolve them into components parallel and perpendicular to OG.
Then,

$$\Sigma\, x\omega^2 \Delta M \cos\theta = 0$$
$$\Sigma\, x\omega^2 \Delta M \sin\theta = 0$$

The resultant of this system therefore reduces to zero.

There now remains the forces $x\alpha\Delta M$. The moment of the force $x\alpha\Delta M$ about G is $x^2\alpha\Delta M$. The summation gives as the total moment,

$$M = \Sigma\, x^2\alpha\Delta M = \alpha\,\Sigma\, x^2\Delta M = I\alpha.$$

Therefore in a rotating body we have two forces F_t and F_r which act through the center of gravity and a moment $I\alpha$. F_r is commonly known as the centrifugal force and sometimes as the radial inertia force. F_t is called the tangential inertia

force. The couple $I\alpha$ may be considered as due to the masses resistance to angular acceleration about its own center of gravity.

The total moment about the center O is the sum of the moment of the couple $I\alpha$ and the moment of F_t. F_r has no moment about O.

$$M_{total} = I\alpha + Mr\alpha \cdot r = \alpha\left[I + Mr^2\right]$$

$$= I'\alpha$$

Therefore to give the body an angular acceleration about O an external moment of $I'\alpha$ is required.

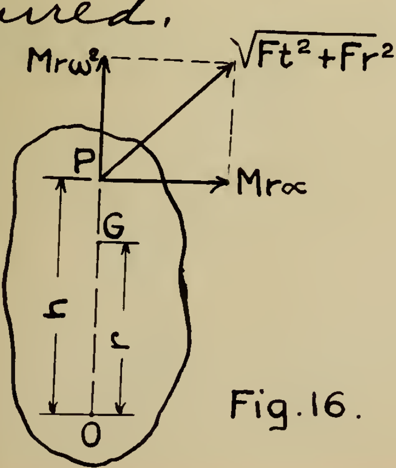

Fig. 16.

F_t, F_r, and $I\alpha$ may be replaced by a single force whose magnitude is $\sqrt{F_r^2+F_t^2}$, see figure 16, and

which acts through a point on the radius drawn through the center of gravity and at a distance $k^2/\bar{r}$. This may be proven as follows:

The total moment about O is $I'\alpha$
Moment of single force is $M\bar{r}\alpha h$
Therefore,

$$M\bar{r}\alpha h = I'\alpha$$

Whence,

$$h = \frac{I'}{M\bar{r}} = \frac{Mk^2}{M\bar{r}} = \frac{k^2}{\bar{r}}.$$

The expressions just deduced may be applied to a body rotating about an instantaneous center.

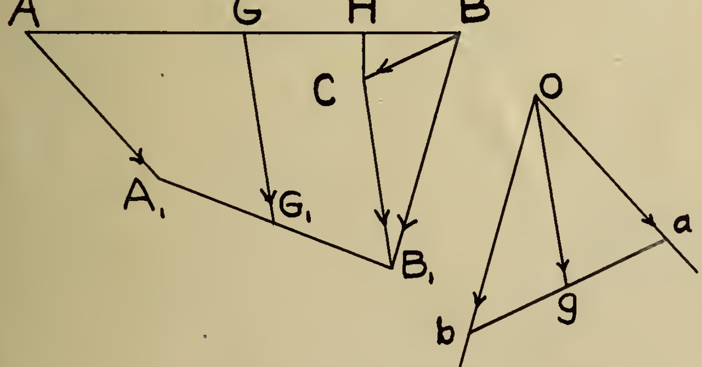

Fig. 17.

In figure 17 let AB be the link of a machine rotating about some instantaneous center. Let AA_1 and BB_1 be accelerations of A and

B respectively. From the acceleration polygon the magnitude and direction of the acceleration of the center of gravity may be found. We may replace the motion of AB by a rotation about G and a translation perpendicular to the line joining G and the instantaneous axis. The acceleration of any point will now be made up of two components, one due to the rotation about G and the other due to translation. G's rotational components are zero therefore G's acceleration is due to translation and is common to all points of the body. Resolve BB_1 into components BC and CB_1 ($=GG_1$), BC is the rotation component. If BC is again resolved into components perpendicular and parallel to AB the tangential acceleration of B, HC, may be obtained.

We have then $\frac{HC}{GB} = \alpha =$ angular acceleration about G. In the preceding paragraph we found that the inertia forces of a rotating body were composed of a single force through the center of gravity and a couple. This force had a direction opposite to that of the total acceleration of the center of gravity and its magnitude was the mass of the body multiplied by the acceleration of the center of gravity. In

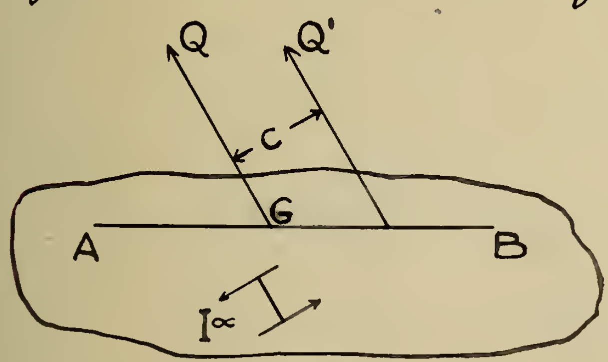

Fig. 18

this case we have therefore a force Q through G and a couple $I\alpha$ as shown in figure 18. The force and couple may be replaced by a single force whose action line is parallel to Q but does

not pass through G. Then a point at a distance c from Q introduce a force Q' equal and parallel to Q. The moment of Q' about G is Q'c.

But $Q'c = I\alpha$

Therefore $c = \dfrac{I\alpha}{Q'}$

This equation gives the distance through which the action line must be shifted from the center of gravity in order to produce the given acceleration.

Q' = mass times acceleration of G and α can be found by method already shown. All terms being known c can be readily calculated.

Another method of finding the action line of the inertia force is to replace the body under consideration by a kinetically equivalent system. Two systems are kinetically equivalent when to give them the

same motion equal translation forces and equal rotation forces are required. The conditions that must be fullfilled therefore are, first, systems must have same mass, second, the same center of gravity, and third, the same moment of inertia. Let origional body have a mass of m gee pounds and the equivalent system be composed of two masses m_1 and m_2 concentrated at the proper points. Applying above condition we have,

1. $m_1 + m_2 = m$ Systems have same mass.

2. $m_1 h_1 = m_2 h_2$ Systems have same center of gravity.

3. $m_1 h_1^2 + m_2 h_2^2 = m k^2$ Systems have same moment of inertia.

Eliminating m_1, m_2, and m from these three equations:

$$k^2 = h_1 h_2$$

K is the mean proportional between h_1 and h_2. Hence if the position of one mass and K be given, the position of the second mass can be found by the following graphical construction.

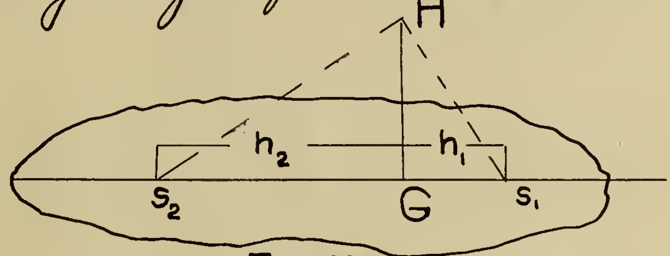

Fig. 19.

In figure 19 let G be the mass center of a body which is symmetrical about the lines S_1S_2. Let S_1 be the given position of one mass. Draw GH perpendicular to S_1S_2 and equal to K. Join H and S_1 and draw HS_2 at right angles to HS_1. S_2 is the position of the second mass.

When the position of the two masses has been located the direction of their accelerations is found. In figure 20 let AB represent some body whose center

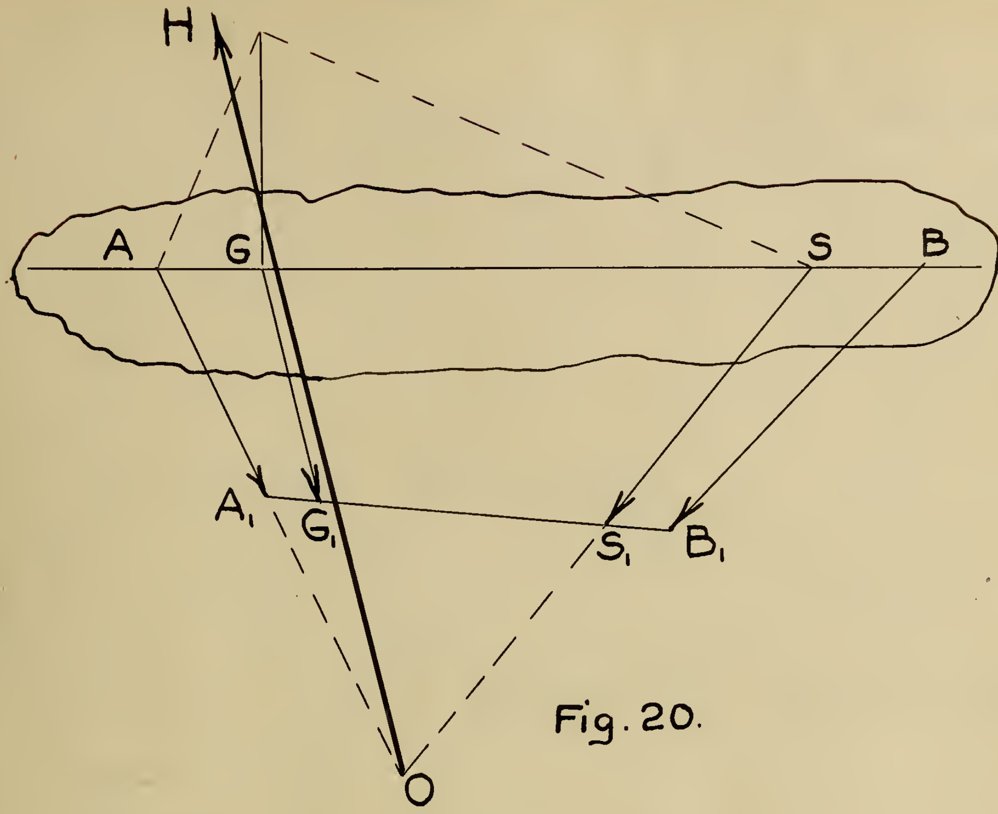

Fig. 20.

of gravity is at G and the acceleration of A and B respectively is AA_1 and BB_1 respectively. The acceleration of any other point as G may now be found, by laying off A_1G_1 proportional to AG. Let the first mass be concentrated at A, then by given construction find S the location of the second mass. Produce the action lines of the accelerating forces of the two masses until they intersect at O. As the inertia force of the

body is the opposite of the resultant accelerating force of the two concentrated masses its action line must pass through O. It has already been shown that the inertia force is parallel to the acceleration of the center of gravity. Therefore through O draw OH parallel to GG_1, this is the action line required.

A still more general case is that in which a point moved in a path which has some plane motion not a translation. An illustration is a point on the pendulum of a shaft governor. This moves in a circle about the center of suspension and the whole circle has a motion about the engine shaft. In this instance the acceleration of the point is not the resultant of the acceleration of the path, and the point in the path, but is instead the resultant of

these two plus a third component which we shall proceed to find.

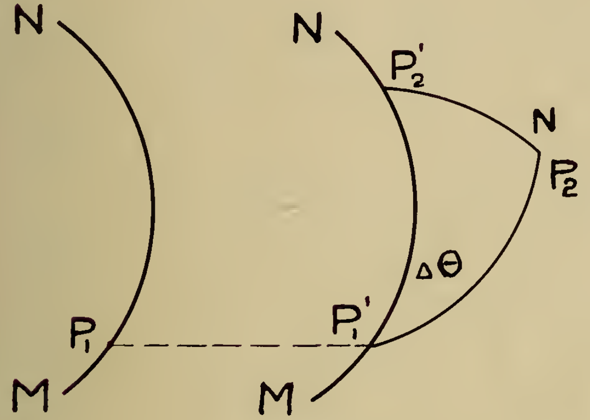

Fig. 21.

In figure 21 the motion of point P from p_1 to its final position p_2 is made in three steps. First a translation of path MN, then motion in the path $P_1'P_2'$, then a rotation of the path through an angle $\Delta\theta$.

Let ΔS = length of path traversed by P.
a = acceleration due to rotation of path.

We have then

$$p_2'p_2 = \frac{1}{2} a \Delta t^2 = \Delta S \cdot \Delta\theta$$

$$a = 2 \frac{\Delta S}{\Delta t} \cdot \frac{\Delta\theta}{\Delta t} \quad \text{approximately.}$$

In the limit therefore

$$a = 2 \frac{ds}{dt} \cdot \frac{d\theta}{dt}.$$

Now $\frac{ds}{dt}$ is the velocity of the point along the curve and is usually denoted by u and $\frac{d\theta}{dt}$ is the angular velocity of the curve and is denoted by ω

Therefore $a = 2u\omega$

The direction of this third component is the limiting direction of P_2P, which is normal to the curve or in other words perpendicular to the relative velocity. The total acceleration of P = acceleration of path plus acceleration of P in the path plus $2u\omega$. This last acceleration is known as the supplementary acceleration component.

The most general application of Coriolis's law is to motion in space. In plane motion, it will be remembered, the motion of the path was replaced by a translation and a rotation about an instantaneous axis perpendicular to the plane of motion. For motion in

space the instantaneous axis is usually inclined at an angle θ with the path. This introduces another factor in the expression for acceleration. To make this application clear a particular example will be taken up. Consider the fly ball governor shown in figure 22.

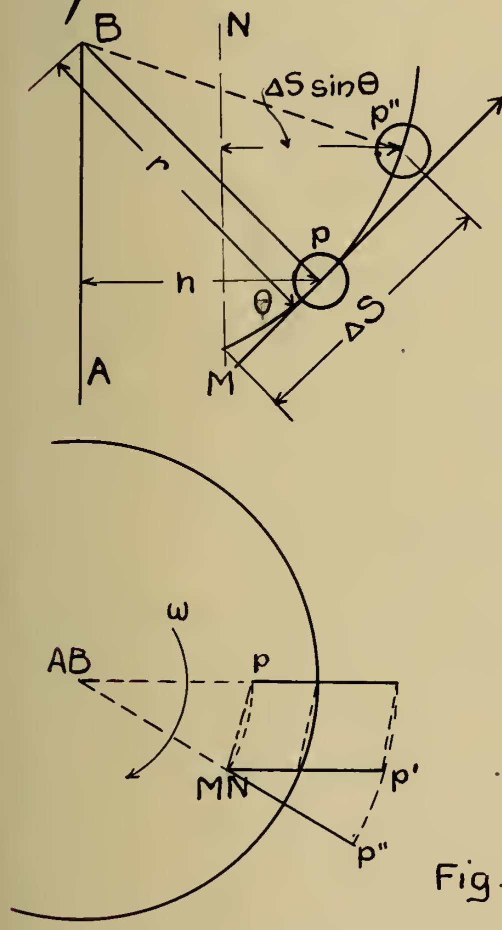

Fig. 22

If the governor changes its configuration, the masses will have an acceleration in the plane of the governor's motion and also an acceleration in a perpendicular plane, about the axis AB. Divide the motion of point P from p to p'' into a rotation about B and a translation to p' and a rotation about the instantaneous axis MN from p' to p''

We now have:

Acceleration of P = acceleration along the path plus acceleration of path plus supplementary acceleration. Supplementary acceleration component = component due to rotation from p' to p''

$$p'p'' = \frac{1}{2} a \Delta t^2 = \Delta\varphi \times \text{radius}$$

But radius $= \Delta s \sin\theta$

Therefore

$$\frac{1}{2} a \Delta t^2 = \Delta\varphi \, \Delta s \sin\theta \quad \text{approximately.}$$

In the limit, therefore

$$a = 2 \frac{ds}{dt} \cdot \frac{d\theta}{dt} \sin\theta$$

$$= 2 u w \sin\theta$$

We now have

$$\text{Acceleration of } P = \frac{du}{dt} + \frac{u^2}{r} + h\alpha + h\omega^2 + 2uw\sin\theta$$

The plus signs indicating geometrical addition.

The components are shown in figure 23.

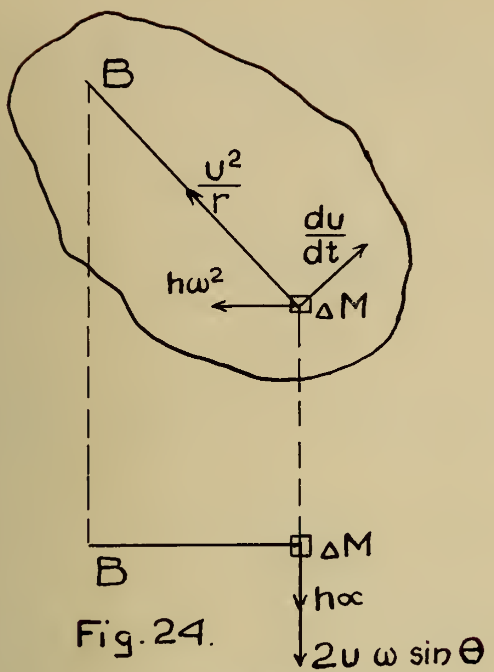

Fig. 24.

Let ΔM denote the particle of mass at P; then the accelerating forces at P are found by multiplying the acceleration components by ΔM. By D'Alembert's principle these forces reversed hold the external forces in equilibrium. We have now to sum each of these for the entire mass. The forces $\Delta M h\alpha$ $\Delta M\, 2u\omega\sin\theta$ and $\Delta M\frac{u^2}{r}$ act either perpendicular to the plane of the governor's motion, or through the center of rotation B. Hence they do no work in moving the governor and may be dropped from further consideration. As the tangential acceleration $\frac{du}{dt}$ is very small the effect of the force $M\frac{du}{dt}$ is negliable, compared with the large centrifugal force which will no be considered.

By summation we get

$$\Sigma\, dF_r = \Sigma \Delta M h \omega^2 = \omega^2 \Sigma \Delta M h$$

But $\Sigma \Delta M h = M h$

Therefore $F_r = M h \omega^2$

Although the forces $2 u \omega \sin\theta \Delta M$ and $h \alpha \Delta M$ act in a plane perpendicular to the governor's motion and do not effect the governoring action yet they exert an influence upon the motion of the engine. These inertia forces must be overcome by the turning moment of the crank shaft, just as the inertia of the fly-wheel and other moving masses must be overcome.

2. Equations for Reducing Forces.

A general equation which can be applied for reducing forces will now be derived. The principal of force reduction has been given as the equality of work. Origional forces will be denoted by Roman capital and the reduced forces by corresponding script capitals. The velocity of the point of application of the original forces will be denoted by v' and the velocity of the point to which they are reduced by v.

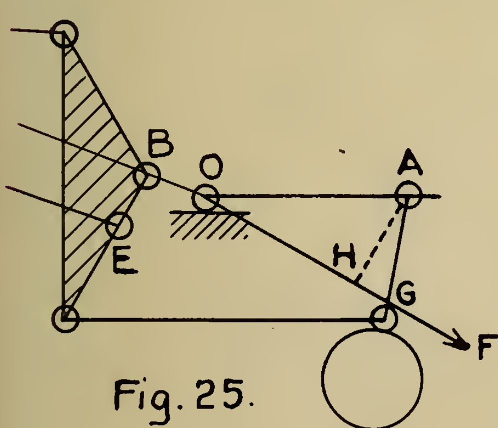

Fig. 25.

Take for illustration the centrifugal shaft governor shown schematically in figure 25. Required to reduce force F to point E, perpendicular to BE. From A drop a perpendicular AH, to action line OG. Take H as the point of application of

force F. To find the work of force F for an instant of time, its magnitude is multiplied by the velocity of H and the product by a differential element of time.

$$\text{Work} = \text{Force} \times ds$$

But $ds = v'dt$

Therefore $W = Fv'dt$

This equation gives the work of the force F for an interval of time dt. The work of the reduced force at E is $\mathcal{F}v\,dt$.

Therefore $Fv'dt = \mathcal{F}v\,dt$

This equation can be used for reducing all of the forces which arise when a mechanism is in motion. The magnitude and method of determining these forces has already been given.

Reduction of the Link Weight considered as a Force. Using our previous equation we have at once

$$\mathcal{G}_n v\,dt = G_n v'dt$$

$$\mathcal{G}_n = \frac{G_n v'}{v}$$

The sign of this force is positive or negative if it helps or hinders the existing motion. Or, in other words, it is positive when it has a component acting in the direction of the path of the center of gravity.

Reduction of the Radial Inertia Force. Let F_r = radial inertia force, then

$$\mathcal{F}_r v\,dt = F_r v'\,dt$$

$$\mathcal{F}_r = \frac{F_r v'}{v}$$

The sign is determined in the same way as in the preceding case.

Reduction of Tangential Inertia Force.
Let F_t = tangential inertia force, then

$$\mathcal{F}_t\, v\, dt = F_t\, v'\, dt$$

$$\mathcal{F}_t = \frac{F_t\, v'}{v}$$

Reduction of Angular Inertia Resultant

Let $I\alpha$ = moment of inertia couple.
ω = angular velocity of link under consideration

Then

$$\mathcal{F}\, v\, dt = I\alpha\, \omega\, dt$$

$$\mathcal{F} = \frac{I\alpha\, \omega}{v}$$

The sign of this force is found from the consideration that each link has an angular acceleration about its own center of gravity. If the inertia couple thus developed helps the existing motion the value of the reduced force is positive, if not the reverse is true.

IV – Applications to Existing Mechanism.

1. Method of determining data, weights etc, tabulated.

The mechanism selected for the application of this theory is the fly-ball governor on the York Ice Machine in the Mechanical Engineering Laboratory. The analysis of this problem has already been given, it remained only to determine the necessary data for a complete solution

The first step taken was to make accurate kinematic drawing of the governor gear in its lowest position. All of the links were then carefully weighed and their centers of gravity located. The moments of inertia of all links having a rotation either about a fixed or an instantaneous axis were then determined by the pendulum methods already described. Data taken from

link No. 1, the fly weight, will serve as an example

Link No. 1

Weight = 11.9 pounds with pin.
Diameter of eye = 5/16 inch.
Center of gravity = 12 inches from center of eye.

1st Observation	– 50 ocillations	in 1 min.	
2nd	– 101	2	.
3rd	– 101	2	.

The following calculations were then made.

$$\text{Mass} = \frac{11.9}{32.2} = .37 \text{ gee pounds.}$$

$$T = \frac{120}{101} = 1.19 \text{ seconds.}$$

$$h = 12\tfrac{5}{16} \text{ inches} = 1.003 \text{ feet.}$$

$$K_s^2 = \frac{T^2 h g}{\pi^2} = \frac{\overline{1.19}^2 \times 1.003 \times 32.2}{4\pi^2} = 1.15$$

$$\bar{K}^2 = K_s^2 - d^2 = 1.15 - \overline{1.003}^2 = .15$$

$$K'^2 = \overline{.15}^2 + \overline{1.0015}^2 = 1.13$$

TABLE I

Link No	T sec.	h	Mass	$K_s'^2$	K'^2	$\bar{K}^2$	I'	$\bar{I}$
1	1.19	1.003	.37	1.15	1.13	.15	.425	.0555
2	.904	.429	.054	.287	.286	.1	.155	.0054
3-4			2.94					
5	.94	.0678	.0777	.0488	.0443	.0438	.00344	
5_1	.6	.151	.0435	.0443	.0323	.0213	.00143	
5_2	calculated		.087					
5_3	1	.628	.386	.514	.469	.129	.181	
$5_3'$	1.27	1.056	.386	1.38	1.28	.26	.495	
5 Total							.186	
5 Total							.5	
6			.087					
7			.053					
8	.656	.151	.123	.052	.039	.029	.0048	

$$\bar{I} = .37 \times .15 = .0555$$

$$I' = .37 \times 1.13 = .425$$

Similar calculations were made for each link and the results tabulated in table I. In this table K_s' is the radius of gyration with respect to the axis of suspension. The determination for link No. 5 was made by separating it into three pieces. This link has an adjustable weight and the moment of inertia was found for two positions of this weight.

2. Method of Reducing Masses.

We can now reduce all the masses of the governor that accumulate kinetic energy to a single mass concentrated at the center of the ball. For example the reduced mass of link No. 1 $= \frac{I'}{r^2} = \frac{.425}{1.17^2} = .31$

Table II contains the calculations made for the first position.

TABLE II

Link No	Mass	I	I′	Formula for Reduced Mass	Reduced Mass
1			.425	$2 \times .425 \times \frac{1}{1.17^2}$	.62
2	.054	.0155	.0505	$2 \times .0505 \times \left[\frac{.75}{.935 \times 1.17}\right]^2$	.0475
3 & 4	2.94			$2.94 \times \left[\frac{.75 \times .875}{.935 \times 1.17}\right]^2$	1.058
5_1			.181	$1.81 \times \left[\frac{.75 \times .875}{.935 \times 1.17 \times .375}\right]^2$	.467
5_2			.495	$.495 \times \left[\frac{.75 \times .875}{.935 \times 1.17 \times .375}\right]^2$	1.27
6	.087				
7	.053			$.14 \times \left[\frac{.75 \times .875 \times .48}{.935 \times 1.17 \times .375}\right]^2$	.0825
8			.0048		
9			.0048	$2 \times .0048 \left[\frac{.75 \times .875 \times .48}{.935 \times 1.17 \times .375 \times .417}\right]^2$	.0326
				Total	2.307
				Total	3.1106

The total range of the governor was divided into five parts and similar calculations were made for each position. The final results are tabulated in table

TABLE III

Position No.	A	B
1	2.307	3.1106
2	2.596	3.541
3	2.926	3.934
4	3.177	4.398
5	3.509	4.89

The values in the column headed A are those obtained by using the smallest value of the moment of inertia of link No. 5. Those in column B by using the larger value.

A graphical representation of the results was made by dropping ordinates from the center of the ball

and laying off on these lines segments proportional to the reduced mass at that point. A smooth curve through these points is called the curve of reduced masses. This curve together with the necessary graphical construction is shown on Plate I on the following page. Curve MN represents values in column A of table and M'N' in column B of the same table. The small circles show the exact positions of the points as found by calculation.

3. Equation of Motion of Fly-ball Governor.

In order to determine the inertia forces arising in a fly-ball governor, it is necessary to know the total acceleration of the flyweight. This requires that the angular acceleration and the velocity of occilation must be known for any governor position, and

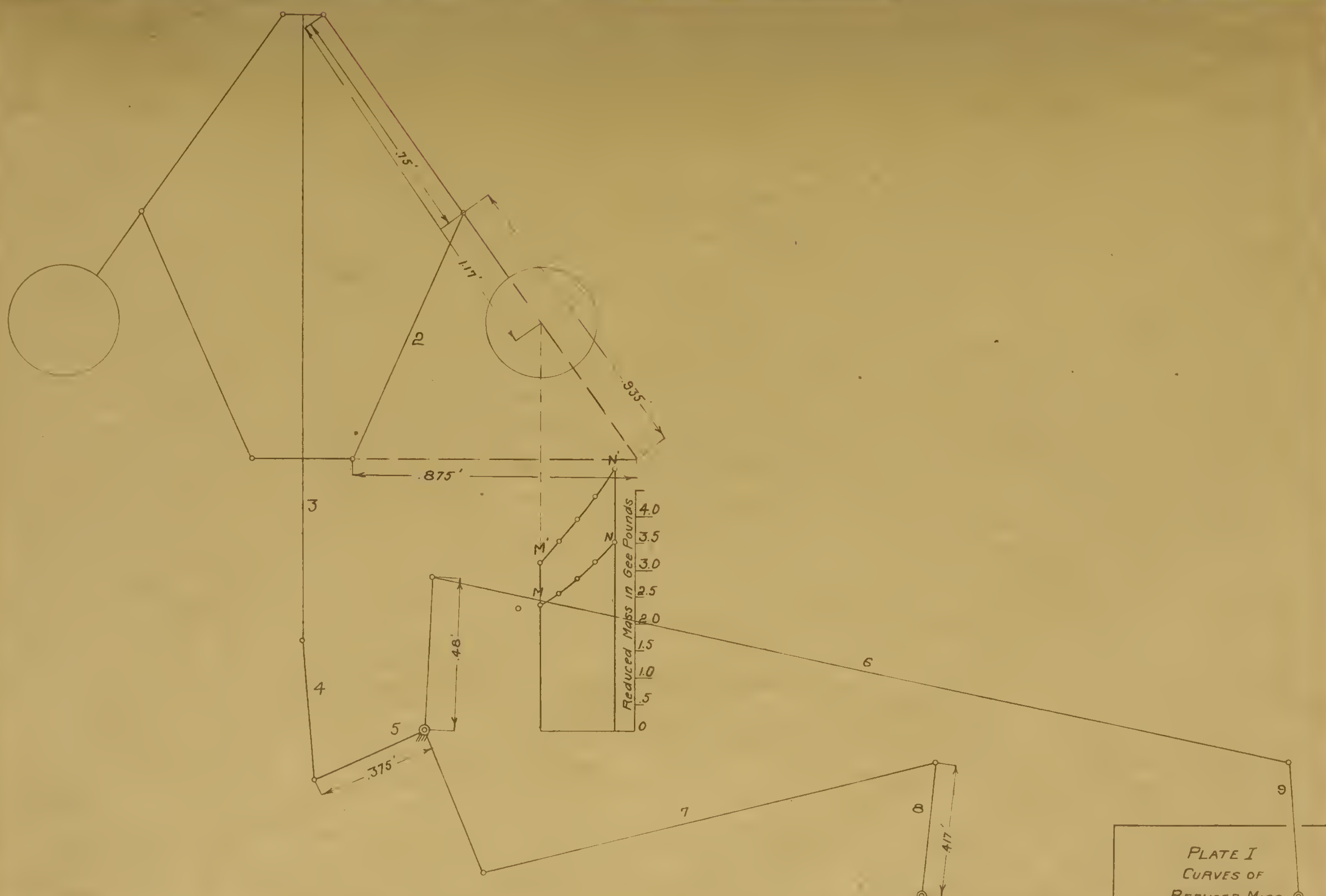

PLATE I
CURVES OF
REDUCED MASS
OF YORK ICE MACH GOV.

for any change of load on the engine. The equation of motion of the fly-ball governor is required for this purpose. The following discussion and the derivation of the equation of motion is due to Tolle.

Before deriving the equation of motion, several other considerations which have a direct bearing upon its development will be taken up. The most important of these is the C curve.

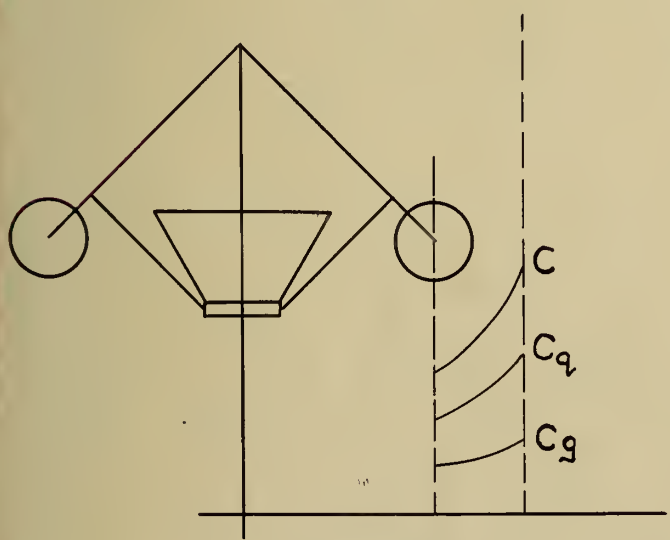

Fig. 26.

From figure 26 it is evident that a certain force is necessary to hold the governor in a given position; this force may be considered to act through the center of gravity of the fly-weight,

and varies with the position chosen. Let ordinates be dropped from the center of gravity of the pendulum to a horizontal line, and distances be laid off vertically to some scale equal to the required force. The points thus found may be connected by a smooth curve, which will be called the C curve. The force thus required may be considered as being made up of two parts, the first C_g, which holds the pendulum in equilibrium and the second C_q which holds the weight. If C represents the total force then

$$C = C_g + C_q$$

The C_g and the C_q curves may be drawn for the entire governor range and the sum of the ordinates at any point will determine a point on the C curve.

The problem of constructing

the C curve now reduces itself to one of graphical statics.

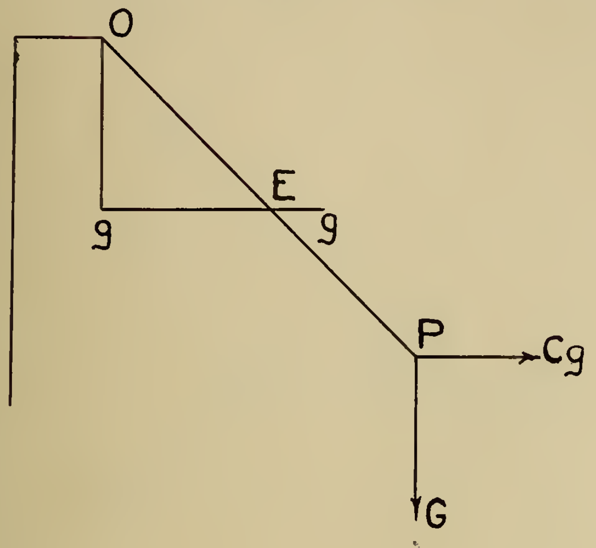

Fig. 27.

If P is the center of gravity of the pendulum and G its weight then the moment of C_g about O must be equal to the moment of about O. From O draw the vertical line Og and make it equal to G, through g draw gg horizontally until it intersects OP at E. gE, then, is the required force for the chosen position of the pendulum.

The force C_q must hold the weight Q in equilibrium. Resolve Q into two components, one along the line BH and the other horizontal. The forces S and C_q must now be in equilibrium with the pin reaction

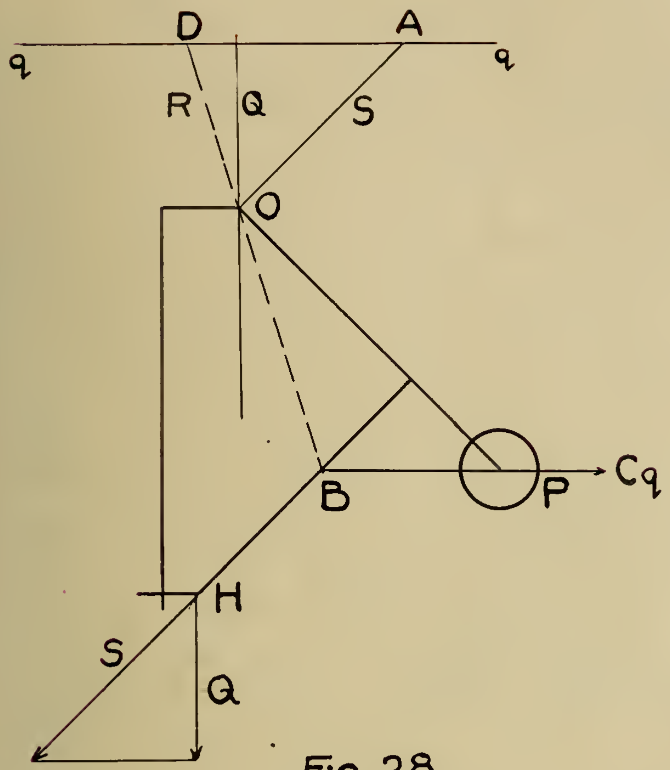

Fig. 28.

at O. If B is the point where C_q intersects the action line of S then BO is the action line of the pin reaction R. Through O draw OA parallel to BH and lay off OA equal to S. Through A draw qq horizontal and produce BO until it intersects qq at D. Then AD gives the value of C_q. This construction lends itself readily to the complete determination of the C_q and the C_q curves. For each position only three new lines need be drawn.

In this discussion friction has not been taken into account.

But for any governor where the size of the pins are known the construction may be modified by taking the pin reactions tangent to the proper friction circles. It is also important to note that the position of the governor axis has not been taken into account in this construction; in other words, the position of the governor axis has no effect on the form of the C curves.

When the governor is in a position of equilibrium, it is held there by centrifugal force due to the rotation of the pendulum about the governor axis. From the foregoing this force C must be equal to $C_g + C_q$.

Let x = distance of center of gravity from axis.

M = mass of pendulum.

n = revolutions per minute.

ω = angular velocity.

Then $\omega = \frac{2\pi n}{60} = \frac{\pi n}{30}$

And $C = Mx\omega^2$ or $\omega^2 = \frac{C}{Mx}$.

Whence $\omega = \sqrt{\frac{C}{x} \cdot \frac{1}{M}}$

Therefore $n = \frac{30\omega}{\pi} = \frac{30}{\pi}\sqrt{\frac{C}{x} \cdot \frac{1}{M}}$

From this equation the revolutions per minute may be calculated when the centrifugal force C and the distance x are known. By drawing separate curves for C_g and C_q the effect of each upon the centrifugal force may be determined.

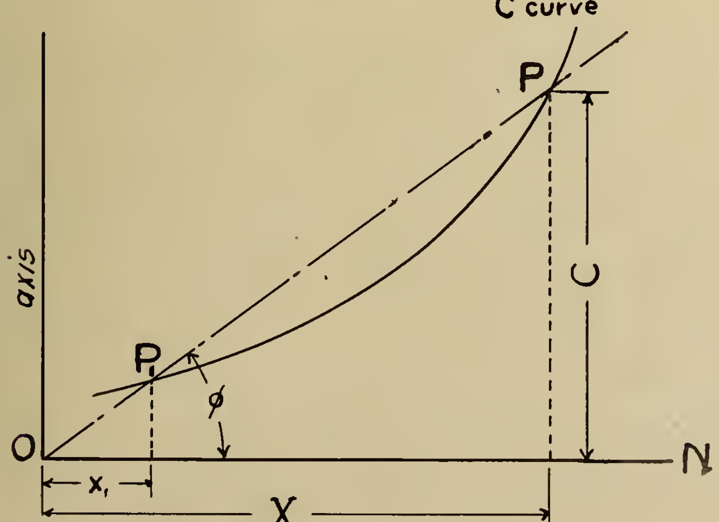

Fig. 29.

In figure 29 a ray is drawn from O to a point P on the C curve. Denoting by ϕ the angle PON

we have: $\frac{c}{x} = \tan\varphi$

Substituting in the equation

$$\omega = \sqrt{\frac{c}{x} \cdot \frac{1}{M}} \quad \text{we get}$$

$$\omega = \sqrt{\frac{1}{M}} \cdot \sqrt{\tan\varphi}$$

And $$n = \frac{30}{\pi}\sqrt{\frac{1}{M}} \cdot \sqrt{\tan\varphi}$$

The form of the C curve illustrated many of the governor's characteristics. If it intersects the ray OP in another point as P_1 then

$$\frac{c_1}{x_1} = \frac{c}{x}$$

and therefore from the preceding equation we have that for both positions of the governor the angular velocity is the same. If the C curve is a straight line through O then the governor has the same angular velocity for all

positions because $\frac{c}{x}$ is constant. Such a governor is called astatic and cannot be used for the regulation of steam engines. It will change its configuration from one extreme position to another with the slightest change in angular velocity. A governor can only be used when it has a different speed for each position: it is then said to be static. If the speed changes the governor will seek its new position and in passing from one to the other it performs its useful function.

Another characteristic of the C curve essential to a good governor should be noted. As the distance x is increased the curve should have such a form that the angle ϕ constantly increases. If this is the case the angular velocity will increase with a rising pendulum. If, on the other hand,

the angle ϕ decreases with increase of the abscissae x then the governor will have the smallest angular velocity for its highest position. It is evident, therefore, that the C curve shows clearly how the angular velocity varies with the governor position. The C curve also distinguishes the character of the governor, if it is astatic or static, useful or not useful. The investigation or design of a governor should therefore begin with the construction of the C curves.

The function of a governor is to keep the speed of the motor constant with any load. This requirement cannot be ridigly fulfilled by a fly-ball governor, because as already shown it must have a certain degree of stability, that is, each position corresponds to a different an-

gular velocity. The increase of angular velocity from the lowest to the highest governor position should therefore be as small as possible. This change in angular velocity is denoted by $\delta\omega$ and δ is called the coefficient of fluctuation. If ω_1 and ω_2 denote the smallest and the largest angular velocities, then $\delta = \frac{\omega_2 - \omega_1}{\omega_m}$ where ω_m is the arithmetical mean of ω_1 and ω_2, that is, $\omega_m = \frac{\omega_2 + \omega_1}{2}$.

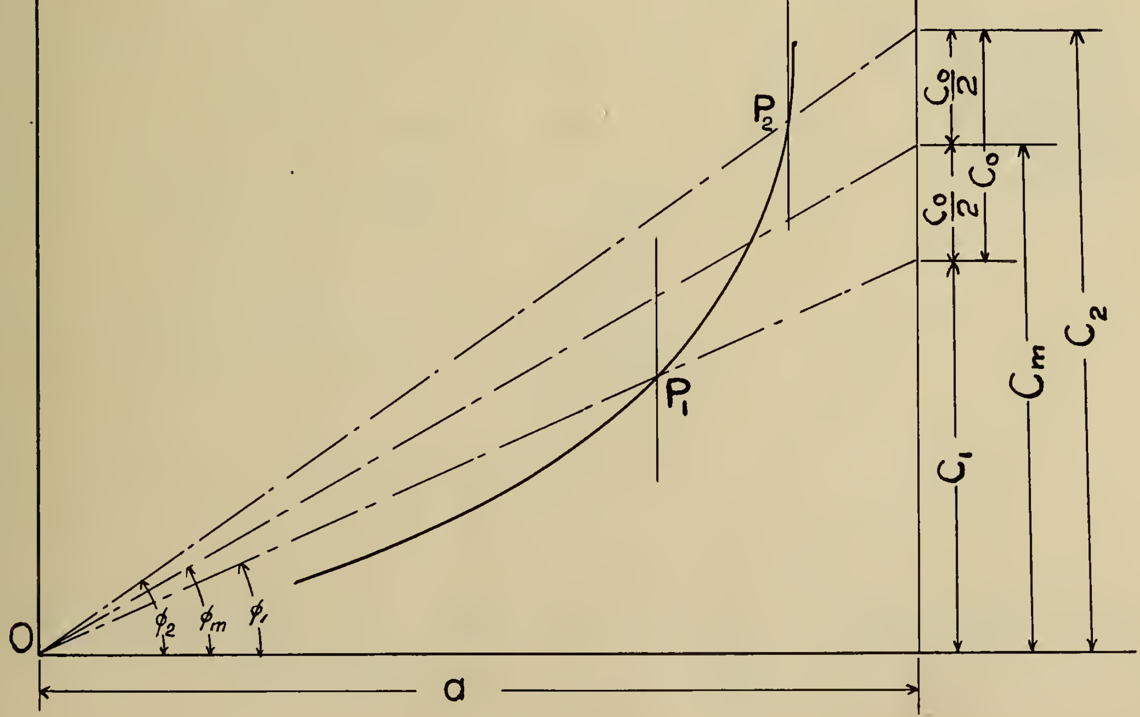

Fig. 30.

$$\delta = \frac{\omega_2 - \omega_1}{\omega_m} = \frac{(\omega_2 - \omega_1)\left(\frac{\omega_2 + \omega_1}{2}\right)}{\omega^2_m} = \frac{\omega_2^2 - \omega_1^2}{2\omega^2_m}$$

But $\omega_2^2 = \frac{1}{M} \tan \varphi_2 \quad \omega_1^2 = \frac{1}{M} \tan \varphi_1$

$$\omega_m^2 = \frac{1}{M} \tan \varphi_m$$

Therefore $\delta = \dfrac{\tan \varphi_2 - \tan \varphi_1}{2 \tan \varphi_m}$

In figure 30 take a equal to the distance of the center of gravity of the pendulum from the axis in its highest position. Through P_1 and P_2 draw rays from O until they intersect a perpendicular through a cutting from it the segments c_1 and c_2

Take $c_m = \dfrac{c_1 + c_2}{2}$

Then $c_0 = c_2 - c_1$

$\tan \varphi_1 = \dfrac{c_1}{a}$, $\tan \varphi_2 = \dfrac{c_2}{a}$, $\tan \varphi_m = \dfrac{c_m}{a}$

Therefore $\delta = \frac{C_2 - C_1}{2C_m} = \frac{C_0}{2C_m}$

or $C_0 = 2\delta C_m$.

If the coefficient of regulation and the extreme governor position is given, C_0 may be calculated and laid off on a perpendicular. If a ray is then drawn through this point it will intersect the C curve in the lowest position. The range of the governor is thus determined.

We will now consider the action of the governor when the load on the motor is changed. As already stated, for each governor position there is a corresponding load on the motor. The lowest position gives the greatest turning moment and the highest the smallest - very nearly zero.- This being the case we may plot a load curve with the position of the center of gravity of the

pendulum as abscissae and load as ordinates. (The ordinates may be expressed as horse power, turning effort or mean effective pressure.)

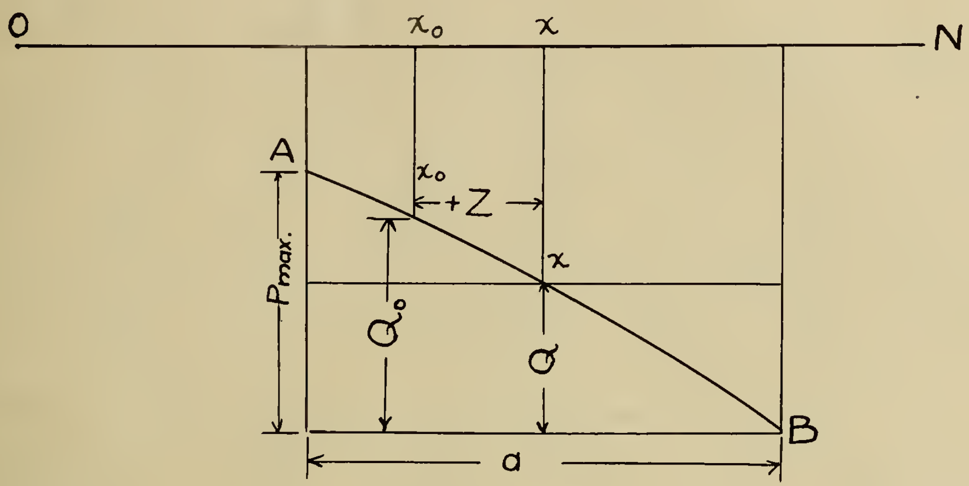

Fig. 31.

Figure 31 shows the resulting load curve AB. This curve will bend slightly at the top for two reasons. First because for the lowest positions a displacement of the pendulum gives smaller displacements along the governor axis than for the extreme higher positions. Secondly, because at large loads a change in the cut-off has less effect on

the mean effective pressure than at light loads. However in order to simplify the discussion the load curve will be assumed to be a straight line.

If the load on the motor is Q_0 the corresponding position of the center of gravity of the governor pendulum is in the axis x_0x_0. Consider now that the load should be suddenly reduced to Q. A disturbance of equilibrium will ensue. As the governor requires some perceptible time to get into its new position xx, there will be an excess turning moment on the engine shaft $Q_0 - Q$. This excess is used in accelerating the shaft and hence also the governor. With an increase in angular velocity there will be an increase in centrifugal force which will shift the governor into its new position. If T is the time in seconds required by the motor

to acquire its normal speed with no load and greatest cut-off, ω the angular velocity and α the angular acceleration of the governor we have:

$$\alpha_{max} = \frac{\omega}{T}$$

This is the maximum angular acceleration of the governor when the load on the motor is suddenly changed from full load to no load.

If the load curve is a straight line then the excess of turning effort over the resisting moment is proportional to z, where z is the distance of the pendulum from the equilibrium position. see figure . Consequently the acceleration of the governor is proportional to z. If a is the range of the governor measured horizontally we have

$$\alpha = \frac{\alpha_{max}}{a} \cdot z = \frac{\omega}{Ta} \cdot z$$

the acceleration for any change in load. The abscissa Z is positive when measured toward the governor axis and negative when measured in an opposite direction

The equation of motion of the fly-ball governor will now be deduced. Let the governor move from its position $x_0 x_0$ to its new position of equilibrium $x x_1$, see figure 32 .

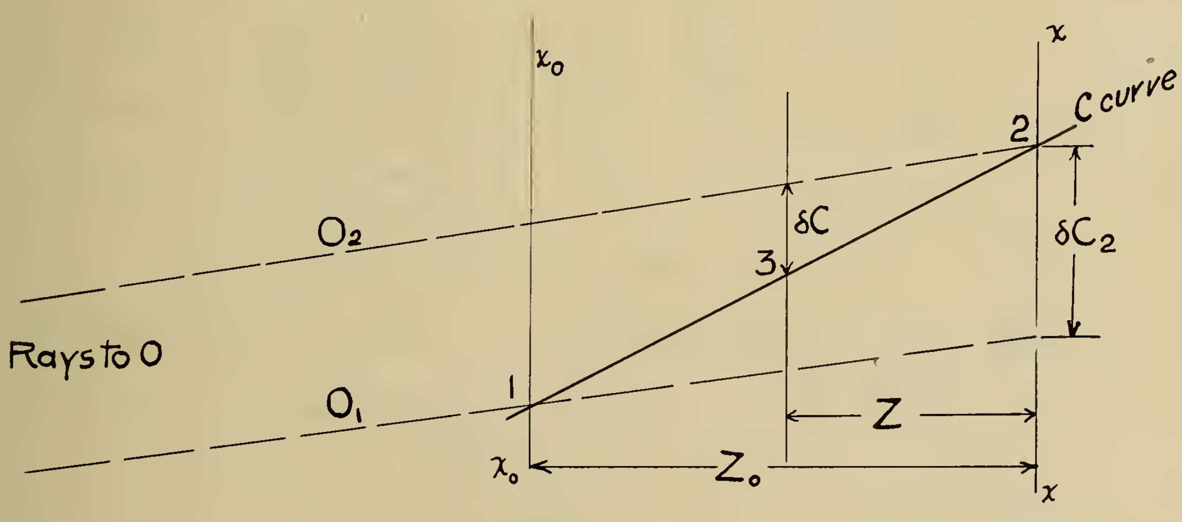

Fig. 32.

We will make the following assumptions.

1. Load curve is a straight line.

2. C curve is a straight line.
3. That the governor range is so small as compared with its distance from the governor axis x that the rays 01 and 02 may be considered parallel.

Also let λ = change in load

Therefore $\lambda = \frac{Z_0}{a}$

If, as soon as equilibrium were disturbed, the angular velocity would rise instantaneously to that corresponding to ray 02 and the center of gravity of the pendulum had only moved to a distance from position 3. A new equilibrium position force δC would be available for accelerating the governor masses. For at the position indicated, the force C measured to the C curve will hold the governor in equilibrium. The excess force δC is proportional to Z. Giving to C_0 its previous value we have

$$\frac{\delta C}{C_0} = \frac{Z}{a} \quad \text{or} \quad \delta C = \frac{C_0}{a} \cdot Z$$

As in reality the rays are not parallel this equation is only approximately true.

But $C_o = 2 C_m \delta$

Therefore $\delta C = \frac{2 \delta \cdot C_m}{a} \cdot Z$

δC is, however not the only force present to accelerate the governor masses. Actually the new angular velocity is not acquired instantaneously. If it takes t seconds for the governor to arrive at position 3 there has been an increase in angular velocity of

$$\Delta\omega = \int \alpha \, dt$$

Hence there is also an increase of centrifugal force δC_1

$$\delta C_1 = (\omega + \Delta\omega)^2 M x - \omega^2 M x = 2\omega \cdot \Delta\omega M x$$

where M is the mass of pendulum

Now let $C_m = M \times \omega^2$

Then $\delta C_1 = 2 \frac{C_m}{\omega} \cdot \Delta\omega$

$$= 2 \frac{C_m}{\omega} \int \alpha \, dt.$$

The new equilibrium position of the governor requires an increase of centrifugal force δC_2

$$\delta C_2 = \frac{C_0}{a} Z_0 = C_0 \lambda = \lambda 2 \delta C_m$$

We have, therefore, the excess of centrifugal force which is available for accelerating the governor masses

$$\delta C_1 - \delta C_2 = \frac{2 C_m}{\omega} \int \alpha \, dt - \lambda 2 \delta C_m$$

The total force which is available for acceleration is

$$P = \delta C + \delta C_1 - \delta C_2 \quad \text{or}$$

$$P = \frac{2 \delta C_m}{a} \cdot Z + \frac{2 C_m}{\omega} \int \alpha \, dt - \lambda 2 \delta C_m$$

But $\alpha = \frac{\omega}{Ta} \cdot z$

Therefore

$$P = \frac{2\delta C m}{a} \cdot z + \frac{2Cm}{Ta}\int z\,dt - \lambda 2 \delta C m$$

Consider all of the masses of the governor gear replaced by a single mass which is moved horizontally by the force P. Let U_r equal the reduced mass and b its acceleration. Then

$$b = \frac{P}{U_r} = -\frac{d^2 z}{dt^2} = \frac{1}{U_r}\left[\frac{2\delta C m}{a} \cdot z + \frac{2Cm}{Ta}\int z\,dt - 2\lambda\delta C m\right.$$

$\frac{d^2z}{dt^2}$ is negative because the acceleration decreases proportional as z increases. Differentiating the preceding equation with respect to t we have

$$\frac{d^3 z}{dt^3} + \frac{2\delta C m}{a U_r}\frac{dz}{dt} + \frac{2Cm}{Ta U_r} \cdot z = 0$$

which is the differential equation of motion of the fly-ball governor. Let $\frac{2Cm}{a U_r} = A$ a constant we have

$$\frac{d^3 z}{d t^3} + \delta A \frac{dz}{dt} + \frac{A}{T} \cdot z = 0 \text{ - - - - - - - - } 1.$$

This equation shows that the velocity and acceleration of the governor pendulum depends upon

1st on $A = \frac{2Cm}{a U_r}$

2nd on δ the coefficient of regulation

3rd on T

The quantity T depends upon the mass of the engine such as the fly wheel, counter weight and other masses which have to be accelerated.

The solution of the differential equation 1 is

$$z = C_1 e^{\omega_1 T} + C_2 e^{\omega_2 T} + C_3 e^{\omega_3 T} \text{ - - - - - - - } 2.$$

Where ω_1, ω_2 and ω_3 are the roots of the cubic equation:

$$\omega^3 + \delta A \omega + \frac{A}{T} = 0 \text{ - - - - - - - - - - - - - - - } 3$$

and C_1, C_2 and C_3 are constants de-

termined by the following conditions

When $t = 0 \quad z = z_0$

$$t = 0 \quad \frac{dz}{dt} = 0$$

$$t = 0 \quad \frac{d^2 z}{dt^2} = 0$$

From Descartes rule of signs equation 3 has one real root $\omega_1 = -2p$ and two imaginary ones $\omega_2 = p + qi$ and $\omega_3 = p - qi$. Substituting these latter values in equation 2 we obtain

$$z = C_1 e^{\omega_1 t} + e^{pt}(C_2 \cos qt + C_3 \sin qt) \text{ - - - - } 4.$$

The roots of equation 3 are

$$\omega_1 = \sqrt[3]{-\frac{A}{2T} + \sqrt{\left(\frac{A}{2T}\right)^2 + \left(\frac{\delta A}{3}\right)^3}}$$

$$+ \sqrt[3]{-\frac{A}{2T} - \sqrt{\left(\frac{A}{2T}\right)^2 + \left(\frac{\delta A}{3}\right)^3}}$$

$$q = \left[\sqrt[3]{-\frac{A}{2T} + \sqrt{\left(\frac{A}{2T}\right)^2 + \left(\frac{\delta A}{3}\right)^3}} \right.$$

$$\left. - \sqrt[3]{-\frac{A}{2T} - \sqrt{\left(\frac{A}{2T}\right)^2 + \left(\frac{\delta A}{3}\right)^3}} \right] \sqrt{3}$$

Applying the three conditions given for determining the constants c_1, c_2 and c_3 to equation 4 and solving we have:

$$C_1 = \frac{p^2 + q^2}{9p^2 + q^2} \cdot Z_0$$

$$C_2 = \frac{8p^2}{9p^2 + q^2} \cdot Z_0$$

$$C_3 = \frac{2q^2 - 6p^2}{9p^2 + q^2} \cdot \frac{p}{q} \cdot Z_0$$

Substituting $-2p$ for w_1 in equation 4 we have

$$Z = C_1 e^{-2pt} + e^{pt}(C_2 \cos qt + C_3 \sin qt)$$

Differentiating, we have, for the velocity.

$$\frac{dZ}{dt} = C_4 e^{-2pt} + e^{pt}(C_5 \cos qt + C_6 \sin qt)$$

where $C_4 = -2pC_1$, $C_5 = pC_2 + qC_3$

$$C_6 = pC_3 - qC_2$$

The acceleration is

$$\frac{d^2 z}{dt^2} = C_7 e^{-2pt} - e^{pt}(C_8 \cos qt + C_9 \sin qt)$$

where $C_7 = C_8 = 4p^2 C_1$, $C_9 = 2pqC_2 + C_3(q^2 - p^2)$

The C curve for the governor was determined experimentally. Simultaneous observations were made of the speed of the governor and the position of the muff. Beginning with the lowest position the speed was increased enough to allow the governor to pass through its entire range. The speed was kept constant in any one position by means of the throttle. Observed values are given in following table, the position of the stem being measured from a fixed point.

Time for 100 R.P.M.	Position of stem
51 sec.	$\frac{15}{16}$"
$50\frac{1}{5}$ "	$1\frac{7}{16}$"
49 "	$1\frac{5}{8}$"
48 "	2"

Now $C = M \times \omega^2 = M \times \left(\frac{2\pi N}{60}\right)^2 = \frac{M \times \pi^2 N^2}{900}$

The distance of the center of gravity of the fly ball from the stem denoted by x was measured from the drawing for each position. All other quantities being known, C can be found.

For position 1, $C = 85$ pounds.

2, $C = 92$

3, $C = 98.6$

4, $C = 106$

The C curve shown on Plate II was plotted with these values. From this curve $C_1 = 85$ pounds $C_2 = 118.5$ pounds.

Therefore $C_m = \frac{85 + 118.5}{2} = 100$ approximately

$$C_0 = C_2 - C_1 = 118.5 - 85 = 33.5 \text{ pounds}$$

$$\delta = \frac{C_0}{2C_m} = \frac{33.5}{2 \times 100} = 16.7\%$$

From the curve of reduced masses, M for the position of greatest cut-off is 3.78 gee pounds. M is the mass reduced to the center of the ball 14 inches from the pin. Reducing M to the center of gravity of the pendulum, 12 inches from the pin we have

$$M_r = \frac{M\,\overline{14}^2}{\overline{12}^2} = \frac{3.78 \times 196}{144} = 5.15 \text{ gee pounds}$$

But M_r is not identical with U_r, because U_r has a velocity $\frac{dz}{dt}$ and M_r a velocity $b\frac{d\theta}{dt}$.
Therefore we have:

$$\frac{1}{2} U_r \left(\frac{dz}{dt}\right)^2 = \frac{1}{2} M_r \left(b\frac{d\theta}{dt}\right)^2$$

But $\frac{dz}{dt} = b\frac{d\theta}{dt} \cos \alpha$

Therefore $U_r = \frac{M_r}{\cos^2\alpha} = \frac{5.15}{\cos^2\alpha} = 9$

U_r may be found graphically when M is known. In figure 33 lay off SC equal to M_r, draw CA perpendicular to SC until it intersects the horizontal AS. From A erect a vertical to B. Then SB represents U_r to the same scale that SC represents M_r.

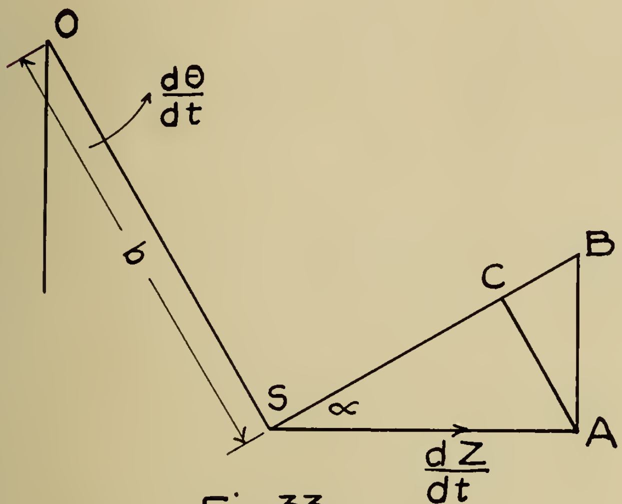

Fig. 33.

The horizontal range of the fly-ball was .115 feet and the time T as observed was two seconds. All quantities involved in the equation are known except Z_o which depends upon the change of load. Assuming that the engine is running at full load and half of the load is suddenly thrown off, we will

consider the following problem. To determine all forces existing in the governor gear at the end of .2 of a second after load has been removed and to reduce these forces to the center of gravity of the fly-ball with a horizontal line of action.

$$A = \frac{2\,C_m}{a\,U_r} = \frac{2 \times 100}{.115 \times 9} = 256$$

$$\delta A = 256 \times .167 = 42.8$$

$$\omega_1 = \sqrt[3]{\frac{256}{2 \times 2} + \sqrt{\left(\frac{256}{4}\right)^2 + \left(\frac{42.8}{3}\right)^3}} + \sqrt[3]{-\frac{256}{4} + \sqrt{\left(\frac{256}{4}\right)^2 + \left(\frac{42.8}{3}\right)^3}}$$

$$\omega_1 = -2.59 \qquad p = 1.29$$

$$q = 8.01 \times \sqrt{3} = 13.9$$

$$C_1 = \frac{\overline{1.29}^2 + \overline{13.9}^2}{+\overline{13.9}^2} \cdot Z_0 = \frac{2.22 \times 194}{20 \times 194} \times .0575 = .0528$$

$$C_2 = \frac{8 \times 2.22}{214} \times .0575 = .00477$$

$$C_3 = \frac{388 - 13.3}{214} \times .0575 \times \frac{1.29}{13.9} = .00108$$

Let $t = .2$

$$Z = \frac{.0528}{e^{.596}} + e^{298}\left(.00477 \cos 13.9 \times .2 + .00108 \sin 13.9 \times .2\right)$$

$$= \frac{.0528}{1.81} + 1.35\left(.00477 \times -.933 + .00108 \times .358\right)$$

$$= .0292 + 1.35\left(-.00445 + .00039\right)$$

$$= .0292 - .00548 = .0237 \text{ feet.}$$

$$C_4 = -2 \times 1.29 \times .0528 = -.1575$$

$$C_5 = 1.29 \times .00477 + 13.9 \times .00108 = .0221$$

$$C_6 = 1.29 \times .00108 - 13.9 \times .00477 = -.06479$$

$$\frac{dZ}{dt} = -\frac{.1575}{1.81} + 1.35\left(-.0221 \times .933 - .06479 \times .358\right)$$

$$= -.087 - .059 = -.146 \text{ feet per second}$$

$$C_7 = C_8 = 4 \times 2.22 \times .0528 = .469$$

$$C_9 = 2 \times 1.49 \times 13.9 \times .00477 + .00108\left(194 - 2.22\right)$$

$$= .198 + .208 = .406$$

$$\frac{d^2 Z}{d t^2} = \frac{.469}{1.81} - 1.35(.469 \times -.933 + .406 \times .358)$$

$= .259 - .394 = -.135$ feet per second per second.

For the change of load let us assume the equilibrium position to be $\frac{a}{2}$ or $\frac{.115}{2} = .0575$ feet from the point M, Plate II,. At the end of .2 of a second $Z = .0237$ feet or the center of gravity of the fly-ball is $.0575 - .0237 = .0358$ feet from M. Lay off MN = .0358, at N erect a perpendicular until it intersects the path of the center of gravity G of the pendulum. The configuration of the governor is now fixed.

The scale of distance is 1 inch equals .166 feet, let scale of velocity be one inch = .09 feet per second. Then the scale of acceleration equals $\frac{.09^2}{.166} = .0486$ feet per second per second. The horizontal component of G's velocity being known we can construct the velocity polygon

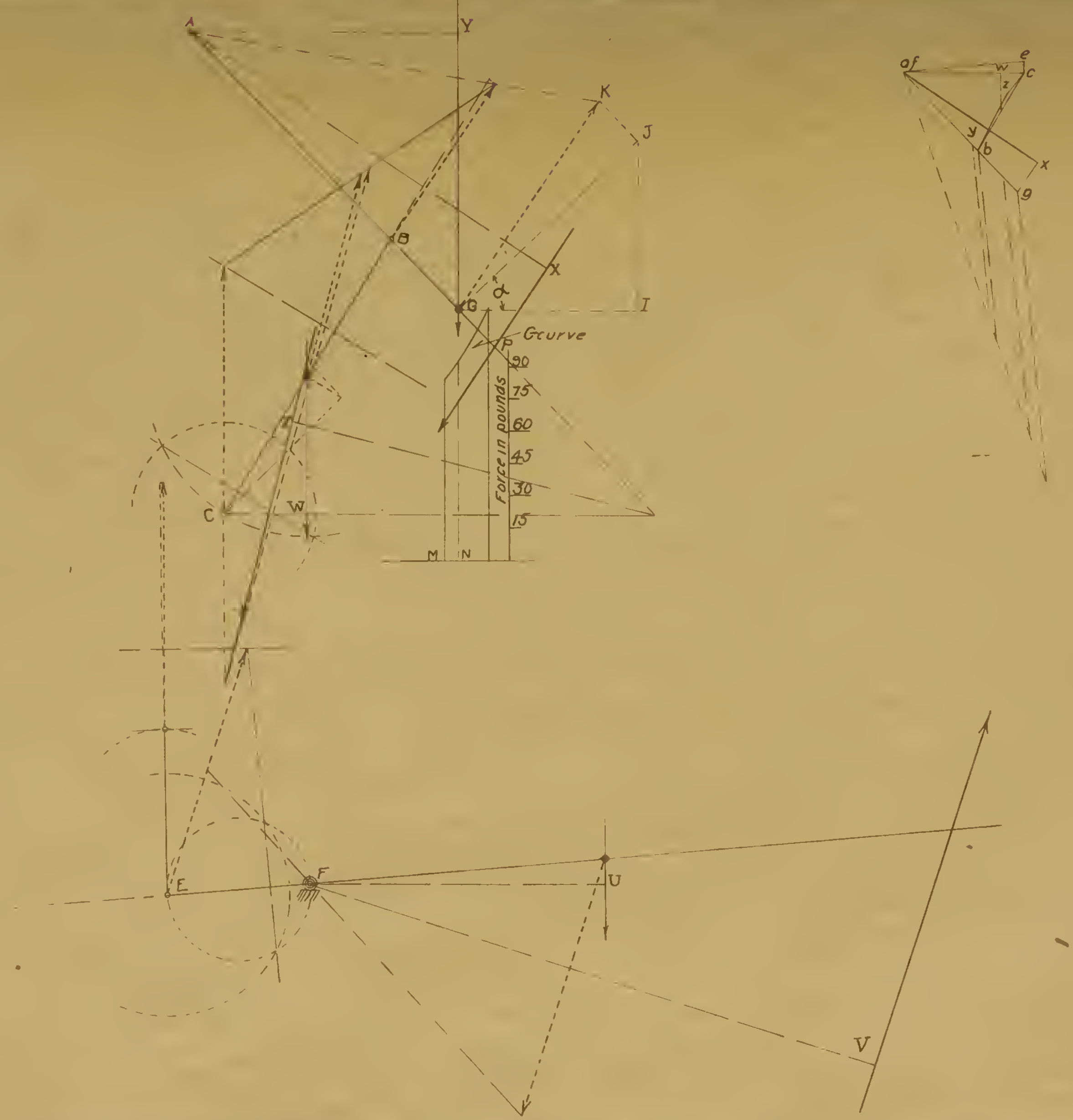

PLATE II
CONSTRUCTIONS
FOR REDUCING FORCES
OF YORK ICE MACH GOV

for the mechanism. Then lay off GI equal to $\frac{d^2Z}{dt^2} = -.146$ feet per second per second. From I erect a perpendicular, IJ, then GJ is G's tangential acceleration. G's radial acceleration $= \frac{\overline{.205}^2}{OG} = \frac{\overline{.205}^2}{1} = .042$ feet per second per second. Lay off JK parallel to AG and equal to .042 feet per second per second, then GK is the total acceleration of G. The construction for the acceleration of the remaining links of the mechanism can now be made. On Plate II, the heavy dotted lines denote accelerations, the heavy solid inertia forces and the light lines the forces due to weight.

The acceleration of G = .19 feet per second per second, mass of link No. 1 is .37 gee pounds.

Therefore $F = .37 \times .19 = .0704$ pounds.

It was shown that this force passed

through a point a distance $\frac{k^2}{r}$ from the center of rotation and is parallel and opposite to G's acceleration

$$\frac{k^2}{r} = \frac{1.13}{1} = 1.13 \text{ feet.}$$

Lay of AP = 1.13 feet and draw XP parallel to G's acceleration, this is the action line of F.

Let $\mathcal{F}$ be the corresponding force reduced to G acting perpendicular to AG. The work of force F = F × velocity of application point X × dt. The work of reduced force $\mathcal{F}$ = $\mathcal{F}$ × velocity of G × dt.

Therefore

$$\mathcal{F} \times \text{velocity of } G \cdot dt = F \times \text{velocity of } X \cdot dt$$

$$\mathcal{F} = \frac{F \times \text{velocity of } X}{\text{velocity of } G}$$

Velocity of X = .228 feet per second.
Velocity of G = .205 feet per second.

$$\mathcal{F} = \frac{.0704 \times .228}{.205} = .0785 \text{ pounds.}$$

The sign of this force is negative because it opposes the governor in its motion. The line of action of the weight is VG and w is the weight reduced we have

$$w = \frac{W \times \text{velocity of } Y}{.205} = \frac{11.9 \times .142}{.205} = 8.28 \text{ lbs.}$$

The sign of this force is also negative because it has a component opposite to the direction of the motion of the center of gravity. The remaining forces were reduced in the same way. The summary of the work is given in table . The summation of the reduced forces is −70.0786 pounds. In obtaining this sum the reduced of links 1 and 2 were multiplied by 2 because there are two of each of these links in the governor mechanism. The action line of these forces was taken perpendicular to AG while the problem called for reduced forces with a horizontal action line. Let

TABLE IV

Link No.	Weight	Mass	Acceleration of Center of Gravity	Inertia Force	Velocity of Application Point	Reduced Force
1		.37	.19 $\frac{ft}{sec^2}$	.0704#	.228 $\frac{ft}{sec}$	−.0785#
1	11.90				.142	−8.28
2		.054	.159	.0086	.151	−.0063
2	1.74				.137	−1.163
3 & 4		2.94	.189	.555	.171	−.463
3 & 4	94.7				.171	−79.
5		.386	.413	.1595	.72	−.56
5	16.4				.36	+28.8

F_n be the required force then

$$F_n = \frac{F}{\cos\alpha} = \frac{-70.0786}{\cos\alpha} = -96 \text{ pounds}$$

The force scaled from the C curve for this position is 94 pounds which is sufficient to hold governor in equilibrium leaving an additional force of 2 pounds to accelerate the governor masses.